On the Correctness of Dominion-Size-Sorting Based Algorithms for the Computation of the Top Cycle in a Tournament Graph

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1 Introduction

Our algorithm for computing the smallest dominant set, the so called Top Cycle (TC) in a tournament graph [1] relies on the TC being a prefix of any descending ordering of the players of that tournament on the size of their dominion.

We write $G = (V, E)$ for a graph, and any graph we discuss is assumed to be a finite, non-empty tournament graph, ie. $1 \leq |V| < \infty$. The dominion of a player $v \in V$ is denoted by $Dom(v)$.

2 Theorem and Proof

Lemma 1. *Let $D \subseteq V$ be a dominant set of G. Then:*

$$\forall d \in D, v \in V \setminus D : |Dom(d)| > |Dom(v)|$$

Proof. The case of $D = \emptyset$ or $D = V$ is trivially true. Otherwise, let $d \in D$ and $v \in V \setminus D$ as above. Since d has defeated, by the definition of dominion, every player in $V \setminus D$:

$$|Dom(d)| \geq |V \setminus D| \tag{1}$$

Because D is a dominion, v cannot have defeated any player in D, otherwise there would be a player in D that has not defeated a player outside D. Additionally, a player cannot have defeated themselves, leaving only the remaining players in $D \setminus V$. Therefore:

$$|Dom(v)| \leq |(V \setminus D) \setminus \{v\}| < |V \setminus D| \tag{2}$$

Combining 1 and 2 gives:

$$|Dom(d)| \overset{(1)}{\geq} |V \setminus D| \overset{(2)}{>} |Dom(v)|$$

Any it therefore follows that:

$$|Dom(d)| > |Dom(v)|$$

□

Theorem 2. *Let $M = (v_1, ..., v_n)$ be a descending ordering of the players of V by the size of their dominion, i.e. $\{v_1, ..., v_n\} = V$ and:*

$$\forall i, j \in \{1, ..., |V|\}, i < j : |Dom(v_i)| \geq |Dom(v_j)| \tag{3}$$

Then every non-empty dominant set $D \subseteq V$ of G is a non-empty prefix of M:

$$\exists i \in \{1, ..., |V|\} : \{v_1, ..., v_i\} = D$$

Proof. Suppose D were a non-empty dominant set of G that were not a prefix of M. Then let $v_i \in D$ be the rightmost vertex with respect to M:

$$\forall j \in \{1, ..., |V|\} : v_j \in D \implies j < i$$

Since D is not a prefix of M, there exists a vertex $v_j \notin D$ left of v_i with respect to M:

$$\exists j \in \{1, ..., |V|\}, j < i : v_j \notin D$$

Since $j < i$, by (3):

$$|Dom(v_j)| \geq |Dom(v_i)|$$

However, by Lemma 1, since $v_i \in D$:

$$v_j \in D$$

which is a contradiction. □

Corollary 2.1. *For any ordering M as in Theorem 2, the Top Cycle TC of G is the shortest non-empty, dominant prefix of M.*

Proof. Suppose $D \neq TC$ were the shortest non-empty, dominant prefix of M. Then, the Top Cycle, itself being a dominant set, must by Theorem 2 also be a prefix of M, and since it may not also be the shortest prefix, it must be a longer prefix. However, then $|D| < |TC|$ which means D is a smaller dominant set than TC, which is a contradiction. □

3 Conclusion

By Theorem 2 and especially Corollary 2.1, we can find the top cycle of a tournament by sorting the players in descending order by the size of their dominions, then checking prefixes in ascending order of length until a dominant set is found.

References

[1] Wikipedia contributors. Tournament (graph theory) — Wikipedia, the free encyclopedia, 2020. [Online; accessed 22-November-2020].