On the decomposition of n-dimensional cuboids into n-dimensional cubes of edge lengths 2^k

Felix Zehetbauer

2020

1 Introduction

We want to find the optimal decomposition of an n-dimensional cuboid into n-dimensional cubes of edge lengths 2^k, $k \in \mathbb{N}_0$. The optimal decomposition is the decomposition that requires the least cubes.[1]

In the following C is an arbitrary n-dimensional cuboid with dimensions $d_1, ..., d_n \in \mathbb{N}_0$. A decomposition $(c_0, ..., c_k)$ of C with $c_0, ..., c_k \in \mathbb{N}_0$ and $c_k \neq 0$ means C can be decomposed into c_i cubes of edge length 2^i.

2 Lemmas and Theorems

Lemma 1. $b_k = \displaystyle\prod_{i=1}^{n} \left\lfloor \frac{d_i}{2^k} \right\rfloor$ *is the number of cubes of edge length* $2^k, k \in \mathbb{N}_0$ *that can fit into* C.

Proof. Let A be an n-dimensional cuboid with dimensions $da_1, ..., da_n \in \mathbb{N}_0$ that is built of cubes of edge length 2^k. Assume, without loss of generality, that $da_i \leq da_{i+1}$ and $d_i \leq d_{i+1}$ for $1 \leq i < n$. A can fit into C, iff $da_i \leq d_i$ for $1 \leq i \leq n$. It is obvious that $da_i = t_i \cdot 2^k$ with t_i being the number of cubes placed next to each other in each dimension. Therefore $\displaystyle\prod_{i=1}^{n} t_i = \prod_{i=1}^{n} \frac{da_i}{2^k}$ is the number of cubes of edge length 2^k that A consists of. Consequently, the largest A built that way, that can fit into C, which has the dimensions $da_i = \left\lfloor \frac{d_i}{2^k} \right\rfloor \cdot 2^k$, consists of $\displaystyle\prod_{i=1}^{n} \left\lfloor \frac{d_i}{2^k} \right\rfloor$ cubes of edge length 2^k. This is also the number of cubes of edge length 2^k that can fit into C because, since $d_i - da_i = d_i - \left\lfloor \frac{d_i}{2^k} \right\rfloor \cdot 2^k = \left(\frac{d_i}{2^k} - \left\lfloor \frac{d_i}{2^k} \right\rfloor\right) \cdot 2^k < 1 \cdot 2^k$, there is obviously no way to fit an additional cube of edge length 2^k into C. $\square$

Corollary 1.1. $b_k = 0, \forall k \geq \lfloor \log_2 (min(d_1, ..., d_n) \rfloor + 1$.

Proof. $0 \leq \lfloor \frac{d_i}{2^k} \rfloor \leq \lfloor \frac{d_i}{2^{\lfloor \log_2 (min(d_1,...,d_n) \rfloor + 1}} \rfloor = 0$ when $d_i = min(d_1, ..., d_n)$.
Therefore $b_k = \prod_{i=1}^{n} \left\lfloor \frac{d_i}{2^k} \right\rfloor = \left\lfloor \frac{d_1}{2^k} \right\rfloor \cdot ... \cdot 0 \cdot ... \cdot \left\lfloor \frac{d_n}{2^k} \right\rfloor = 0$. □

Lemma 2. *(b_0) is a valid decomposition of C.*

Proof. According to Lemma 1, the cube of edge length 2^0 fits b_0 times into C and because the volume of a cube of edge length 2^0 $V_{c_1} = 1$, it follows that $b_0 \cdot V_{c_1} = b_0 = \prod_{i=1}^{n} \left\lfloor \frac{d_i}{2^0} \right\rfloor = \prod_{i=1}^{n} d_i = V_C$. Therefore (b_0) is a valid decomposition of C. □

Theorem 3. *$dec(C) \coloneqq (b_0 - b_1 \cdot 2^n, b_1 - b_2 \cdot 2^n, ..., b_k)$ is a valid decomposition of C.*

Proof. According to Lemma 1, every n-dimensional cube of edge length 2^i fits 2^n times into a cube of edge length 2^{i+1}. In addition, the volume of 2^n cubes of edge length 2^i is the same as the volume of a cube of edge length 2^{i+1}. Therefore in a decomposition, every cube of edge length 2^{i+1} can be replaced by 2^n cubes of edge length 2^i. Hence $b_i - b_{i+1} \cdot 2^n$ calculates how many cubes of edge length 2^i would fit into the cuboid, excluding the area, which could be occupied by cubes of edge length 2^{i+1}. That means $b_i - b_{i+1} \cdot 2^n$ is the number of cubes of edge length 2^i, which cannot be replaced by a bigger cube. The existence of a valid decomposition $(b_0 - b_1 \cdot 2^n, ..., b_i)$ implies that $(b_0 - b_1 \cdot 2^n, ..., b_i - b_{i+1} \cdot 2^n, b_{i+1})$ is also a valid decomposition (given that $b_{i+1} \neq 0$). Because of that, Lemma 2 implies that dec(C) is a valid decomposition. □

Theorem 4. *$dec(C)$ is the optimal decomposition of C.*

Proof. Let t_i be defined as the number of cubes of edge length 2^i in a decomposition of C. Every valid decomposition $(t_0, ..., t_k)$ of C has a maximum of $b_i - t_{i+1} \cdot 2^n - t_{i+2} \cdot 2^{2 \cdot n} - ... - t_k \cdot 2^{(k-i) \cdot n}$ cubes of edge length 2^i because, according to Lemma 1, every cube of edge length 2^{i+j} consists of $2^{j \cdot n}$ cubes of edge length 2^i and can therefore be replaced by them. Let $\Delta_i \geq 0$ be the number of cubes of edge length 2^i a decomposition has less than maximum possible. Hence t_i can be written as

$$
\begin{aligned}
t_i &= b_i - t_{i+1} \cdot 2^n - t_{i+2} \cdot 2^{2 \cdot n} - ... - t_k \cdot 2^{(k-i) \cdot n} - \Delta_i \\
&= b_i - \overbrace{(b_{i+1} - t_{i+2} \cdot 2^n - ... - t_k \cdot 2^{(k-(i+1)) \cdot n} - \Delta_{i+1})}^{t_{i+1}} \cdot 2^n \\
&\quad - t_{i+2} \cdot 2^{2 \cdot n} - ... - t_k \cdot 2^{(k-i) \cdot n} - \Delta_i \\
&= b_i - (b_{i+1} - \Delta_{i+1}) \cdot 2^n + (t_{i+2} \cdot 2^{2 \cdot n} + ... + t_k \cdot 2^{(k-i) \cdot n}) \\
&\quad - (t_{i+2} \cdot 2^{2 \cdot n} + ... + t_k \cdot 2^{(k-i) \cdot n}) - \Delta_i \\
&= b_i - (b_{i+1} - \Delta_{i+1}) \cdot 2^n - \Delta_i.
\end{aligned}
$$

It can be assumed that $\Delta_0 = 0$ because every decomposition with $\Delta_0 \neq 0$ is obviously invalid. Let $f(\Delta_1, ..., \Delta_k)$ be the number of cubes of the decomposition $(t_0, ..., t_k)$ of C. Hence

$$\begin{aligned} f(\Delta_1, ..., \Delta_k) &= t_0 + t_1 + ... + t_k \\ &= (b_0 - (b_1 - \Delta_1) \cdot 2^n) + (b_1 - (b_2 - \Delta_2) \cdot 2^n - \Delta_1) \\ &\quad + ... + (b_k - \Delta_k) \\ &= \overbrace{b_0 + (b_1 + ... + b_k) \cdot (1 - 2^n)}^{\text{const.}} \\ &\quad + (\Delta_1 + ... + \Delta_k) \cdot \underbrace{(2^n - 1)}_{> 0}. \end{aligned}$$

$f(\Delta_1, ..., \Delta_k)$ is minimal exactly when $\Delta_1 + ... + \Delta_k$ is minimal. This is the case when $\Delta_1 = ... = \Delta_k = 0$. Therefore if the decomposition $(t_0, ..., t_k)$ with $t_i = b_i - b_{i+1} \cdot 2^n$ is valid, it must be the optimal one. Because this is how $dec(C)$, which is according to Theorem 3 a valid decomposition, was defined, $dec(C)$ must be the optimal decomposition of C. $\square$

3 Conclusion

With the application of these Lemmas and Theorems, it is possible to implement an efficient algorithm to decompose an n-dimensional cuboid C into n-dimensional cubes of edge lengths 2^k, $0 \leq k \leq \lfloor \log_2 (min(d_1, ..., d_n) \rfloor$ that needs according to Corollary 1.1 $\mathcal{O}(n \cdot \log (min(d_1, ..., d_n)))$ arithmetic operations.

References

[1] Prof. Tobias Nipkow, Ph.D.,
J. Rädle, L. Stevens, K. Kappelmann, MC Sr Eberl:
Functional Programming and Verification Sheet 5,
`https://www21.in.tum.de/teaching/fpv/WS20/assets/ex05.pdf`