

Exercise 1 (Confluence of β -Reduction)

In the lecture we have shown the confluence of \longrightarrow_β using the diamond property of parallel β -reduction. In this exercise, we develop an alternative proof.

We define the operation $*$ on λ -terms inductively over the structure of terms:

$$\begin{aligned} x^* &= x \\ (\lambda x. t)^* &= \lambda x. t^* \\ (t_1 t_2)^* &= t_1^* t_2^* \quad \text{if } t_1 t_2 \text{ is not } \beta\text{-reducible.} \\ ((\lambda x. t_1) t_2)^* &= t_1^*[t_2^*/x] \end{aligned}$$

- Show that we have for two arbitrary λ -terms s and t : $s > t \implies t > s^*$
- Show that \longrightarrow_β is confluent.

Solution

- The proof of

$$s > t \implies t > s^*$$

is done by structural induction on s . In the following, we will skip instances of $s > s$, as we can easily prove $s > s^*$ by another structural induction on s .

1st case: $s = x$.

Let $x > t$. Hence $t = x$. Thus $t = x > x = x^* = s^*$.

2nd case: $s = \lambda x. s_1$.

Let $\lambda x. s_1 > t$. Hence $t = \lambda x. t_1$ for an appropriate t_1 that can be obtained from the inductive definition of $>$, and we have $s_1 > t_1$. As s_1 is a subterm of s , the induction hypothesis implies $t_1 > s_1^*$. Thus we get $\lambda x. t_1 > \lambda x. s_1^* = s^*$.

3rd case: $s = s_1 s_2$, but no β -redex.

Let $s_1 s_2 > t$. Then $t = t_1 t_2$ with $s_1 > t_1$ and $s_2 > t_2$ (again by the inductive definition of $>$) and the induction hypothesis implies $t_1 t_2 > s_1^* s_2^* = s^*$.

4th case: $s = (\lambda x. s_1) s_2$.

Let $(\lambda x. s_1) s_2 > t$. Then there is t_1, t_2 with $s_1 > t_1$ and $s_2 > t_2$ and we have $t = (\lambda x. t_1) t_2$ or $t = t_1[t_2/x]$, depending on the rule used to derive $s > t$.

Case 4.1 $t = (\lambda x.t_1)t_2$.

By induction hypothesis, we get $t > s_1^*[s_2^*/x] = s^*$.

Case 4.2 $t = t_1[t_2/x]$.

By induction hypothesis, and the substitution property of $>$ (Lemma 1.2.13), we get $t = t_1[t_2/x] > s_1^*[s_2^*/x] = s^*$.

b) Part a) immediately implies the diamond property of $>$: Let $s > t_1$ and $s > t_2$. Then we have both, $t_1 > s^*$ and $t_2 > s^*$.

Exercise 2 (Parallel Beta Reduction)

Show:

$$s > t \implies s \rightarrow_{\beta}^* t$$

Solution

Lemma 1.2.11 from the lecture.

Exercise 3 (Predecessor and Tail)

- Define a predecessor function `pred` on church numerals.
- Use the same idea to define `tl` on the list encoding from homework 2.5.

Solution

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$$\begin{aligned} \text{zz} &= \text{pair } \underline{0} \ \underline{0} \\ \text{ss} &= \lambda p.\text{pair}(\text{snd } p)(\text{succ}(\text{snd } p)) \\ \text{pred} &= \lambda m.\text{fst}(m \ \text{ss} \ \text{zz}) \end{aligned}$$

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$$\begin{aligned} \text{nxt} &= \lambda c.\lambda xp.\text{pair}(\text{snd } p)(c \ x \ (\text{snd } p)) \\ \text{tl} &= \lambda l.\lambda cn.\text{fst} \ (l \ (\text{nxt } c)(\text{pair } n \ n)) \end{aligned}$$

Homework 4 (Parallel Beta Reduction & Substitution)

Show:

$$s > s' \wedge t > t' \implies s[t/x] > s'[t'/x]$$

Homework 5 (Equivalence modulo β -conversion)

Assume that we add the additional axiom

$$\lambda xy.x =_{\beta} \lambda xy.y$$

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- a) Show that under this assumption $t =_{\beta} t'$ for all t, t' .
- b) Repeat the same for the axiom $\lambda x.x =_{\beta} \lambda xy.yx$.

Homework 6 (Böhm's Theorem)

Böhm's Theorem states that for arbitrary closed terms $M \neq N$ without constant atoms in $\beta\eta$ -normal form, there exist $n \geq 0$ and L_1, \dots, L_n such that:

$$M L_1 \dots L_n x y \rightarrow_{\beta}^* x \text{ and } N L_1 \dots L_n x y \rightarrow_{\beta}^* y.$$

That is, we can tell M and N apart. Show the following two special cases:

- a) $M = \lambda xyz.xz(yz)$ and $N = \lambda xyz.x(yz)$
- b) $M = \lambda xy.x(yy)$ and $N = \lambda xy.x(yx)$