Technische Universität München
Institut für Informatik
Prof. Tobias Nipkow, Ph.D.
Simon Wimmer

Lambda Calculus
Winter Term 2017/18
Solutions to Exercise Sheet 5

Exercise 1 (Confluence & Commutation)

Show: If $\rightarrow_1$ and $\rightarrow_2$ are confluent, and if $\rightarrow_1^*$ and $\rightarrow_2^*$ commute, then $\rightarrow_{12} := \rightarrow_1 \cup \rightarrow_2$ is also confluent.

Solution

Lemma A.3.2 from the lecture. The key idea is to consider $\rightarrow_1^* \circ \rightarrow_2^*$ as $\rightarrow_{12}^*$ unfolds into iterations of this relation. More precisely:

$$\rightarrow_{12} \subseteq \rightarrow_1^* \circ \rightarrow_2^* \subseteq \rightarrow_{12}^* \qquad (*)$$

It is easy to see that $\rightarrow_1^* \circ \rightarrow_2^*$ has the diamond property (see picture in the lecture notes). Thus $\rightarrow_1^* \circ \rightarrow_2^*$ is strongly confluent, and together with the "sandwich" property for $\rightarrow_1^* \circ \rightarrow_2^*$ and $\rightarrow_{12}$, we get that $\rightarrow_{12}$ is confluent.

Exercise 2 (Local Commutation)

Show: If

$$t_2 \ {}_2\!\leftarrow s \rightarrow_1 t_1 \Longrightarrow \exists u.\, t_2 \rightarrow_1^= u \ {}_2^*\!\leftarrow t_1 \,,$$

then $\rightarrow_1^*$ and $\rightarrow_2^*$ commute.

Here $\rightarrow^=$ denotes the reflexive closure of $\rightarrow$, i.e.:

$$\rightarrow^= := \rightarrow \cup \rightarrow^0$$

Solution

Lemma A.3.3 from the lecture.

Exercise 3 (Strong Confluence)

A relation $\rightarrow$ is said to be *strongly confluent* iff:

$$t_2 \leftarrow s \rightarrow t_1 \Longrightarrow \exists u.\, t_2 \rightarrow^* u \ {}^=\!\leftarrow t_1$$

Show that every *strongly confluent* relation is also *confluent*.

Solution

We show that every strongly confluent relation is also semi-confluent (see homework). To do so, we will show the stronger property

$$t_2 \ {}^{n}\!\!\leftarrow s \rightarrow t_1 \Longrightarrow \exists u.\, t_2 \rightarrow^* u \ {}^{=}\!\!\leftarrow t_1$$

by induction on n. The base case for $n = 0$ is trivial (choose $u = t_1$). For the induction step, we assume the statement for some u as the induction hypothesis and assume another step $t_2 \rightarrow t'_2$. We make a case distinction on $t_2 \rightarrow^= u$. The case $t_2 = u$ is trivial as then $s \rightarrow^{n+1} t'_2$. If $t_2 \rightarrow u$, then from strong confluence we obtain a u' such that

$$u \rightarrow^* u' \wedge t'_2 \rightarrow^= u'$$

Together with the induction hypothesis, this concludes the proof.

Homework 4 (Semi-Confluence)

A relation $\rightarrow$ is said to be *semi-confluent* iff:

$$t_2 \ {}^{*}\!\!\leftarrow s \rightarrow t_1 \Longrightarrow \exists u.\, t_2 \rightarrow^* u \ {}^{*}\!\!\leftarrow t_1$$

Show that $\rightarrow$ is *semi-confluent* if and only if it is *confluent*.

Homework 5 (Diamond Property & Normal Forms)

Show that if $\rightarrow$ has the diamond property, every element is either in normal form or has no normal form.

Homework 6 (Weak Diamond Property)

Assume that $\rightarrow$ has the following weaker diamond property:

$$t_2 \leftarrow s \rightarrow t_1 \wedge t_1 \neq t_2 \Longrightarrow \exists u.\, t_2 \rightarrow u \leftarrow t_1.$$

a) Is it still the case that every element is either in normal form or has no normal form?

b) Show that if t has a normal form, then all its reductions to its normal form have the same length.