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Lambda Calculus
Winter Term 2017/18
Solutions to Exercise Sheet 11

Exercise 1 (Peirce's Law in Intuitionistic Logic)

Prove the following variant of Peirce's Law in inuitionistic logic:

$$((((P \to Q) \to P) \to P) \to Q) \to Q$$

Solution

Let $A_3 = (P \to Q) \to P$, $A_2 = A_3 \to P$, and $A_1 = A_2 \to Q$.

$$
\begin{prooftree}
\AxiomC{$A_1, A_3, P \vdash A_1$}
\AxiomC{$A_1, A_3, P \vdash P$}
\RightLabel{$\to$I}
\UnaryInfC{$A_1, A_3, P \vdash A_2$}
\RightLabel{$\to$E}
\BinaryInfC{$A_1, A_3, P \vdash Q$}
\RightLabel{$\to$I}
\UnaryInfC{$A_1, A_3 \vdash P \to Q$}
\AxiomC{$A_1, A_3 \vdash A_3$}
\RightLabel{$\to$E}
\BinaryInfC{$A_1, A_3 \vdash P$}
\RightLabel{$\to$I}
\UnaryInfC{$A_1 \vdash A_2$}
\AxiomC{$A_1 \vdash A_1$}
\RightLabel{$\to$E}
\BinaryInfC{$A_1 \vdash Q$}
\RightLabel{$\to$I}
\UnaryInfC{$\vdash ((((P \to Q) \to P) \to P) \to Q) \to Q$}
\end{prooftree}
$$

Exercise 2 (Intuitionistic Proof Search in Haskell)

The goal of this exercise is to implement the procedure to decide $\Gamma \vdash A$ in Haskell, i.e. the algorithm from the proof of Theorem 4.0.6.

- Have a look at the template provided on the website. It provides definitions of formulae and proof terms of intuitionistic propositional logic.
- Try to fill in the implementation of *solve*.
- Implement the three proof rules seen in the lecture: *assumption*, *intro*, and *elim*. Use the examples at the end of the template to test your implementation as you go. For *elim*, use the criterion from the proof to guess suitable instantiations.

Solution

See `prover.hs`.

Homework 3 (Constructive Logic)

a) Prove the following statement using the calculus for intuitionistic propositional logic:

$$((c \to b) \to b) \to (c \to a) \to ((a \to b) \to b)$$

Hint: To make your proof tree more compact, you may remove unneeded assumptions to the left of the $\vdash$ during the proof as you see fit. For example, the following step is valid:

$$\frac{p \vdash p}{p, q \vdash p}$$

b) Give a well-typed expression in $\lambda^{\to}$ with the type

$$((\gamma \to \beta) \to \beta) \to (\gamma \to \alpha) \to ((\alpha \to \beta) \to \beta)$$

(You don't need to give the derivation tree.)

Homework 4 (The Negative Fragment)

We say that a formula A is negative if atomic formulas P only occur *negated* in A, i.e. in the form $P \to \bot$ ($\neg P$ for short). The symbol $\bot$ for *falsehood* plays the rule of an unprovable propositional constant: we do not have any special proof rules or axioms for it.

Show that if A is negative, then:

$$\vdash \neg\neg A \to A$$

Hint: First show:

a) $\vdash \neg\neg\neg A \to \neg A$

b) $\vdash \neg\neg(A \to B) \to (\neg\neg A \to \neg\neg B)$

c) $\vdash (\neg\neg A \to \neg\neg B) \to (A \to \neg\neg B)$