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Lambda Calculus
Winter Term 2021/22
Solutions to Exercise Sheet 4

Exercise 1 (Confluence of β-Reduction)

In the lecture we have shown the confluence of $\longrightarrow_\beta$ using the diamond property of parallel β-reduction. In this exercise, we develop an alternative proof.

We define the operation $*$ on λ-terms inductively over the structure of terms:

$$\begin{aligned} x^* &= x \\ (\lambda x.\ t)^* &= \lambda x.\ t^* \\ (t_1\ t_2)^* &= t_1^*\ t_2^* \qquad \text{if } t_1\ t_2 \text{ is not a } \beta\text{-redex.} \\ ((\lambda x.\ t_1)\ t_2)^* &= t_1^*[t_2^*/x] \end{aligned}$$

a) Show that we have for two arbitrary λ-terms s and t: $s > t \implies t > s^*$

b) Show that $\longrightarrow_\beta$ is confluent.

Solution

a) The proof of

$$s > t \quad\implies\quad t > s^*$$

is done by structural induction on s. In the following, we will skip instances of $s > s$, as we can easily prove $s > s^*$ by another structural induction on s.

1st case: $s = x$.

Let $x > t$. Hence $t = x$. Thus $t = x > x = x^* = s^*$.

2nd case: $s = \lambda x.\ s_1$.

Let $\lambda x.\ s_1 > t$. By case analysis on the derivation of $s > t$ we obtain $t = \lambda x.\ t_1$ for an appropriate t_1 and we have $s_1 > t_1$. As s_1 is a subterm of s, the induction hypothesis implies $t_1 > s_1^*$. Thus we get $t = \lambda x.\ t_1 > \lambda x.\ s_1^* = s^*$.

3rd case: $s = s_1\ s_2$, but no s is not a β-redex.

Let $s_1\ s_2 > t$. Then $t = t_1\ t_2$ with $s_1 > t_1$ and $s_2 > t_2$ (again this follows by case analysis using the inductive definition of $>$). We get the induction hypotheses $t_1 > s_1^*$ and $t_2 > s_2^*$. By the definition of $>$ this implies $t_1\ t_2 > s_1^*\ s_2^* = s^*$.

4th case: $s = (\lambda x.\ s_1)\ s_2$.

Let $(\lambda x.\ s_1)\ s_2 > t$. Then there is t_1, t_2 with $s_1 > t_1$ and $s_2 > t_2$ and we have $t = (\lambda x.\ t_1)\ t_2$ or $t = t_1[t_2/x]$, depending on the rule used to derive $s > t$. Note that the first case actually requires a nested case distinction: first we consider the case that $t = t'_1\ t_2$ by the third rule in the definition of $>$. This requires that $(\lambda x.\ s_1) > t'_1$. Hence, $t'_1 = (\lambda x.\ t_1)$ and $s_1 > t_1$ follows by another case distinction on $(\lambda x.\ s_1) > t'_1$.

Case 4.1 $t = (\lambda x.\ t_1)\ t_2$.

By the induction hypotheses $t_1 > s_1^*$ and $t_2 > s_2^*$, we get $t = (\lambda x.\ t_1)\ t_2 > s_1^*[s_2^*/x] = s^*$.

Case 4.2 $t = t_1[t_2/x]$.

By the induction hypotheses $t_1 > s_1^*$ and $t_2 > s_2^*$, and the substitution propery of $>$ (Lemma 1.2.13), we get $t = t_1[t_2/x] > s_1^*[s_2^*/x] = s^*$.

b) Part a) immediately implies the diamond property of $>$: Let $s > t_1$ and $s > t_2$. Then we have both, $t_1 > s^*$ and $t_2 > s^*$. Like in the lecture we obtain the confluence of $\longrightarrow_\beta$ using the diamond property.

Exercise 2 (Parallel Beta Reduction)

Show:

$$s > t \Longrightarrow s \longrightarrow_\beta^* t$$

Solution

Proof by rule induction on $>$.

Case $s > s$**:** We have that $s = t$ and $s \longrightarrow_\beta^* s$.

Case $\lambda x.\ s > \lambda x.\ s'$**:** As an induction hypothesis we get $s \longrightarrow_\beta^* s'$. By Lemma 1.2.3 it then holds that $\lambda x.\ s \longrightarrow_\beta^* \lambda x.\ s'$. Consequently, we have $\lambda x.\ s \longrightarrow_\beta^* \lambda x.\ s'$.

Case $s\ t > s'\ t'$**:** As induction hypotheses we get $s \longrightarrow_\beta^* s'$ and $t \longrightarrow_\beta^* t'$. With Lemma 1.2.3 we get $s\ t \longrightarrow_\beta^* s'\ t \longrightarrow_\beta^* s'\ t'$ and therefore $s\ t \longrightarrow_\beta^* s'\ t'$.

Case $(\lambda x.\ s)\ t > s'[t'/x]$**:** As induction hypotheses we get $s \longrightarrow_\beta^* s'$ and $t \longrightarrow_\beta^* t'$. With Lemma 1.2.3 it holds that $(\lambda x.\ s)\ t \longrightarrow_\beta^* (\lambda x.\ s')\ t \longrightarrow_\beta^* (\lambda x.\ s')\ t' \longrightarrow_\beta s'[t'/x]$ and thus $(\lambda x.\ s)\ t \longrightarrow_\beta^* (\lambda x.\ s')\ t'$.

Exercise 3 (Predecessor and Tail)

a) Define a predecessor function pred on church numerals.

b) Use the same idea to define tl on the list encoding from homework 2.5.

Solution

a)

$$\begin{aligned}
\mathsf{zz} &= \ \mathsf{pair}\ \underline{0}\ \underline{0} \\
\mathsf{ss} &= \lambda p.\ \mathsf{pair}\ (\mathsf{snd}\ p)\ (\mathsf{succ}\ (\mathsf{snd}\ p)) \\
\mathsf{pred} &= \lambda m.\ \mathsf{fst}\ (m\ \mathsf{ss}\ \mathsf{zz})
\end{aligned}$$

b)

$$\begin{aligned}
\mathsf{nxt} &= \ \lambda c.\ \lambda x\ p.\ \mathsf{pair}\ (\mathsf{snd}\ p)\ (c\ x\ (\mathsf{snd}\ p)) \\
\mathsf{tl} &= \ \lambda l.\ \lambda c\ n.\ \mathsf{fst}\ (l\ (\mathsf{nxt}\ c)\ (\mathsf{pair}\ n\ n))
\end{aligned}$$

Homework 4 (Parallel Beta Reduction & Substitution)

Show:

$$s > s' \wedge t > t' \Longrightarrow s[t/x] > s'[t'/x]$$

Homework 5 (Equivalence modulo β-conversion)

Assume that we add the additional axiom

$$\lambda x\ y.\ x =_\beta \lambda x\ y.\ y$$

.

a) Show that under this assumption $t =_\beta t'$ for all t, t'.

b) Repeat the same for the axiom $\lambda x.\ x =_\beta \lambda x\ y.\ y\ x$.

Homework 6 (Böhm's Theorem)

Böhm's Theorem states that for arbitrary closed terms $M \neq N$ without constant atoms in $\beta\eta$-normal form, there exist $n \geq 0$ and $L_1, \ldots, L_n$ such that:

$$M\ L_1\ \ldots\ L_n\ x\ y \rightarrow_\beta^* x \text{ and } N\ L_1\ \ldots\ L_n\ x\ y \rightarrow_\beta^* y\,.$$

That is, we can tell M and N apart. Show the following two special cases:

a) $M = \lambda x\ y\ z.\ x\ z\ (y\ z)$ and $N = \lambda x\ y\ z.\ x\ (y\ z)$

b) $M = \lambda x\ y.\ x\ (y\ y)$ and $N = \lambda x\ y.\ x\ (y\ x)$