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Lambda Calculus
Winter Term 2021/22
Solutions to Exercise Sheet 7

Exercise 1 (Progress Property)

Let t be a closed and well-typed term, i.e. $[] \vdash t : \tau$ for some τ. Show that t is either a value or there is a t' such that $t \rightarrow_{cbv} t'$.

Solution

The proof follows an induction on the derivation of $[] \vdash t : \tau$. The variable case cannot occur. Abstractions are values, so there is nothing to do here. For the application case, assume $t = t_1\, t_2$ and $[] \vdash t_1 : \tau_1 \rightarrow \tau_2$ and $[] \vdash t_2 : \tau_1$. By the induction hypothesis, both, t_1 and t_2, can take a step or are a value. If t_1 can take a step, we can use the left application rule on t. If t_1 is a value and t_2 can take a step, then the right application rule can be used. If t_1 and t_2 are both values, we know $t_1 = \lambda x.\ t_1'$ for some t_1' as t_1 is of type $\tau_1 \rightarrow \tau_2$. Thus we can apply the rule for reducing an abstraction.

Exercise 2 (Normal Form)

Show that every type-correct $\lambda^{\rightarrow}$-term has a β-normal form.

Solution

We regard a reduction strategy that is guaranteed to terminate. The strategy is chosen such that it decreases the types of subterms.

Let $|\tau|$ be the size of a type τ, i.e. the number of function-arrows occuring in τ.

$$|\alpha| = 0$$
$$|\alpha \rightarrow \beta| = |\alpha| + |\beta| + 1$$

With this measure, we can assign a natural number to each β-redex:

$$|(\lambda x.\ \ s)\ t| = |\tau_1 \rightarrow \tau_2| \quad \text{where} \quad [] \vdash \lambda x.\ \ s : \tau_1 \rightarrow \tau_2$$

Thus, for any term t, we obtain a multiset

$$M_t = \{|\,(\lambda x.\ \ s)\ t' \mid \exists\, x\ s\ t'.\ \ (\lambda x.\ \ s)\ t' \text{ is a subterm of } t\,|\}$$

M_t is the multiset of the sizes of all β-redexes in t.

We can view multisets as functions into the natural numbers and define an ordering on them:

$$M <_M N \text{ iff}$$
$$M \neq N \wedge$$
$$\forall y.\ \ M(y) > N(y) \Longrightarrow \exists x.\ \ y < x \wedge M(x) < N(x)$$

It can be proved that the multiset ordering terminates (is well-founded).

If one regards a β-redex of the form $r = (\lambda x.\ u)\ v$ with u and v in β-NF, then we have for the reduct $r' = u[v/x]$: $M_r >_M M_{r'}$. This is because of $[] \vdash \lambda x.\ u \colon \tau_1 \to \tau_2$ and $[] \vdash v \colon \tau_1$, the substitution may create new β-redexes w, but for all those w in r' we have $|w| < |r|$:

> w is of the form $(v\ v')$ with $v = (\lambda x.\ t)$ for some t since $v \in \mathsf{NF}$
> and thus $|w| = |\tau_1| < |\tau_1 \to \tau_2| = |r|$.

Thus, if we choose a reduction strategy $\to_p$ that reduces an innermost β-redex in t, we have:

$$t \to_p t' \Longrightarrow M_t >_M M_{t'}$$

We can obtain such a reduction strategy by restricting the first rule of $\to_\beta$ to:

$$\frac{s \in \mathrm{NF} \qquad t \in \mathrm{NF}}{(\lambda x.\ s)\, t \to_p s[t/x]}$$

As the multiset ordering terminates, also the chosen reduction strategy must terminate. If it terminates with t', then t' is in β-NF.

$\square$

Homework 3 (Typing)

a) Prove:

$$[] \vdash (\lambda x\colon \tau_2 \to \tau_3.\ \lambda y\colon \tau_1 \to \tau_2.\ \lambda z\colon \tau_1.\ x\,(y\,z))\colon (\tau_2 \to \tau_3) \to (\tau_1 \to \tau_2) \to \tau_1 \to \tau_3$$

b) Give suitable solutions for $?\tau_1$, $?\tau_2$, $?\tau_3$ and $?\tau_4$ and prove that the term is type-correct given your solution.

$$[] \vdash \lambda x :?\tau_1.\ \lambda y :?\tau_2.\ \lambda z :?\tau_3.\ \ x\ y\ (y\ z)\ :?\tau_4$$

Homework 4 (β-reduction preserves types)

A type system has the *subject reduction property* if evaluating an expression preserves its type. Prove that the simply typed λ-calculus ($\lambda^{\to}$) has the subject reduction property:

$$\Gamma \vdash t\colon \tau \wedge t \to_\beta t' \Longrightarrow \Gamma \vdash t'\colon \tau$$

Hints: Use induction over the inductive definition of $\to_\beta$ (Def. 1.2.2). State your inductive hypotheses precisely – it may help to introduce a binary predicate $P(t, t')$ to express the property you are proving by induction. Also note that the proof will require *rule inversion*: Given $\Gamma \vdash t\colon \tau$, the shape of t (variable, application, or λ-abstraction) may determine which typing rule must have been used to derive the typing judgment.

Within your proof, you are free to use the following lemma about substitution:

$$\Gamma \vdash u\colon \tau_0 \wedge \Gamma[x\colon \tau_0] \vdash t\colon \tau \Longrightarrow \Gamma \vdash t[u/x]\colon \tau \qquad (1)$$

Homework 5 (Implementation of multiset-ordering and reduction)

Implement the multiset ordering and the reduction strategy from the second tutorial exercise in your favorite programming language.