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Lambda Calculus
Winter Term 2022/23
Exercise Sheet 5

Exercise 1 (Confluence & Commutation)

Show: If $\to_1$ and $\to_2$ are confluent, and if $\to_1^*$ and $\to_2^*$ commute, then $\to_{12} := \to_1 \cup \to_2$ is also confluent.

Exercise 2 (Strong Confluence)

A relation $\to$ is said to be *strongly confluent* iff:

$$t_2 \leftarrow s \to t_1 \Longrightarrow \exists u.\ t_2 \to^= u \ {}^*\!\!\leftarrow t_1$$

Show that every *strongly confluent* relation is also *confluent*.

Exercise 3 (Local Confluence of η-reduction)

Analogously to β-reduction, we define η-reduction inductively:

1. $x \notin \mathsf{FV}(s) \Longrightarrow (\lambda x.\ s\ x) \to_\eta s$
2. $s \to_\eta s' \Longrightarrow s\ t \to_\eta s'\ t$
3. $s \to_\eta s' \Longrightarrow t\ s \to_\eta t\ s'$
4. $s \to_\eta s' \Longrightarrow (\lambda x.\ s) \to_\eta (\lambda x.\ s')$

The proof of local confluence of $\to_\eta$, i.e. it holds that there exists a u with $t_1 \to_\eta^* u \ {}_\eta^*\!\!\leftarrow t_2$ if we have $t_1 \ {}_\eta\!\!\leftarrow s \to_\eta t_2$, was very informal. Give a proper proof using this definition.

Homework 4 (Semi-Confluence)

A relation $\rightarrow$ is said to be *semi-confluent* iff:

$$t_2 \ {}^*\!\!\leftarrow s \rightarrow t_1 \Longrightarrow \exists u.\ t_2 \rightarrow^* u \ {}^*\!\!\leftarrow t_1$$

Show that $\rightarrow$ is *semi-confluent* if and only if it is *confluent*.

Homework 5 (Diamond Property & Normal Forms)

Show that if $\rightarrow$ has the diamond property, every element is either in normal form or has no normal form.

Homework 6 (Weak Diamond Property)

Assume that $\rightarrow$ has the following weaker diamond property:

$$t_2 \leftarrow s \rightarrow t_1 \land t_1 \neq t_2 \Longrightarrow \exists u.\ t_2 \rightarrow u \leftarrow t_1.$$

a) Is it still the case that every element is either in normal form or has no normal form?

b) Show that if t has a normal form, then all its reductions to its normal form have the same length.