

Exercise 1 (Peirce's Law in Intuitionistic Logic)

Prove the following variant of Peirce's Law in intuitionistic logic:

$$(((P \rightarrow Q) \rightarrow P) \rightarrow P) \rightarrow Q \rightarrow Q$$

Solution

Let $A_3 = (P \rightarrow Q) \rightarrow P$, $A_2 = A_3 \rightarrow P$, and $A_1 = A_2 \rightarrow Q$.

$$\begin{array}{c}
 \frac{A_1, A_3, P \vdash A_1 \quad \frac{A_1, A_3, P \vdash P}{A_1, A_3, P \vdash A_2} \rightarrow I}{A_1, A_3, P \vdash Q} \rightarrow E \\
 \frac{A_1, A_3, P \vdash Q}{A_1, A_3 \vdash P \rightarrow Q} \rightarrow I \quad \frac{A_1, A_3 \vdash A_3}{A_1, A_3 \vdash P} \rightarrow E \\
 \frac{A_1, A_3 \vdash P \rightarrow Q \quad A_1, A_3 \vdash P}{A_1 \vdash A_2} \rightarrow I \\
 \frac{A_1 \vdash A_2 \quad A_1 \vdash A_1}{A_1 \vdash Q} \rightarrow E \\
 \frac{A_1 \vdash Q}{\vdash (((P \rightarrow Q) \rightarrow P) \rightarrow P) \rightarrow Q \rightarrow Q} \rightarrow I
 \end{array}$$

Exercise 2 (Intuitionistic Proof Search in Haskell)

The goal of this exercise is to implement the procedure to decide $\Gamma \vdash A$ in Haskell, i.e. the algorithm from the proof of Theorem 4.1.2.

- Have a look at the template provided on the website. It provides definitions of formulae and proof terms of intuitionistic propositional logic.
- Try to fill in the implementation of *solve*.
- Implement the three proof rules seen in the lecture: *assumption*, *intro*, and *elim*. Use the examples at the end of the template to test your implementation as you go. For *elim*, use the criterion from the proof to guess suitable instantiations.

The algorithm can be streamlined further:

- When trying to prove $\Gamma \vdash A \rightarrow B$, it suffices to try (\rightarrow Intro). Explain why.

- b) The attempt to prove $\Gamma \vdash A$ by assumption can be dropped if we use the following generalised \rightarrow Elim rule:

$$\frac{\Gamma \vdash A_1 \rightarrow \dots \rightarrow A_n \rightarrow B \quad \forall i \leq n. \Gamma \vdash A_i}{\Gamma \vdash B} \rightarrow\text{ELIM}$$

However, the proof obtained can be different. Explain the difference and why the outright proof by assumption is subsumed.

Solution

See `prover.hs` for the implementation.

In the following we will denote the by (\rightarrow Elim) the more general rule described in lemma 4.1.2.

- a) Suppose we prove $\Gamma \vdash A \rightarrow B$ by an application of (\rightarrow Elim). The proof will be of the following format:

$$\frac{\Gamma \vdash A_1 \rightarrow \dots \rightarrow A_n \rightarrow A \rightarrow B \quad \forall i \leq n. \Gamma \vdash A_i}{\Gamma \vdash A \rightarrow B} \rightarrow\text{ELIM}$$

We can always provide an alternative proof that uses (\rightarrow Intro) first and looks like this:

$$\frac{\frac{\Gamma \vdash A_1 \rightarrow \dots \rightarrow A_n \rightarrow A \rightarrow B \quad \forall i \leq n. \Gamma, A \vdash A_i \quad \Gamma, A \vdash A}{\Gamma, A \vdash B} \rightarrow\text{ELIM}}{\Gamma \vdash A \rightarrow B} \rightarrow\text{INTRO}$$

The case where $\Gamma \vdash A \rightarrow B$ is proved by assumption is subsumed by the next answer.

- b) Proof by assumption is just a special case of (\rightarrow Elim) where $n = 0$. However, if we drop the assumption rule, proofs can now have a slightly different structure because we try (\rightarrow Intro) first:

$$\frac{\frac{A_1 \rightarrow \dots \rightarrow A_n \rightarrow B \in \Gamma' \quad \forall i \leq n. \Gamma' \vdash A_i}{\Gamma' \vdash B} \rightarrow\text{ELIM}}{\Gamma, A_1 \rightarrow \dots \rightarrow A_n \rightarrow B \vdash A_1 \rightarrow \dots \rightarrow A_n \rightarrow B} \rightarrow\text{INTRO } n \text{ TIMES}$$

with

$$\Gamma' := \Gamma, A_1 \rightarrow \dots \rightarrow A_n \rightarrow B, A_1, \dots, A_n.$$

Homework 3 (Constructive Logic)

a) Prove the following statement using the calculus for intuitionistic propositional logic:

$$((c \rightarrow b) \rightarrow b) \rightarrow (c \rightarrow a) \rightarrow ((a \rightarrow b) \rightarrow b)$$

Hint: To make your proof tree more compact, you may remove unneeded assumptions to the left of the \vdash during the proof as you see fit. For example, the following step is valid:

$$\frac{p \vdash p}{p, q \vdash p}$$

b) Give a well-typed expression in λ^{\rightarrow} with the type

$$((\gamma \rightarrow \beta) \rightarrow \beta) \rightarrow (\gamma \rightarrow \alpha) \rightarrow ((\alpha \rightarrow \beta) \rightarrow \beta)$$

(You don't need to give the derivation tree.)

Homework 4 (The Negative Fragment)

We say that a formula A is negative if atomic formulas P only occur *negated* in A , i.e. in the form $P \rightarrow \perp$ ($\neg P$ for short). The symbol \perp for *falsehood* plays the role of an unprovable propositional constant: we do not have any special proof rules or axioms for it.

Show that if A is negative, then:

$$\vdash \neg\neg A \rightarrow A$$

Hint: First show:

- a) $\vdash \neg\neg\neg A \rightarrow \neg A$
- b) $\vdash \neg\neg(A \rightarrow B) \rightarrow (\neg\neg A \rightarrow \neg\neg B)$
- c) $\vdash (\neg\neg A \rightarrow \neg\neg B) \rightarrow (A \rightarrow \neg\neg B)$