Technische Universität München
Institut für Informatik
Prof. Tobias Nipkow, Ph.D.
Lukas Stevens

Lambda Calculus
Winter Term 2023/24
Solutions to Exercise Sheet 4

Exercise 1 (β-reduction on de Bruijn Preserves Substitution)

We consider an alternative representation of λ-terms that is due to de Bruijn. In this representation, λ-terms are defined according to the following grammar:

$$d ::= i \in \mathbb{N}_0 \mid d_1\ d_2 \mid \lambda\ d$$

Remember that we defined substitution for de Bruijn terms as follows:

$$\begin{aligned} i \uparrow_l &= \begin{cases} i, & \text{if } i < l \\ i+1, & \text{if } i \geq l \end{cases} \\ (d_1\ d_2) \uparrow_l &= d_1 \uparrow_l\ \ d_2 \uparrow_l \\ (\lambda\ d) \uparrow_l &= \lambda\ d \uparrow_{l+1} \end{aligned}$$

$$\begin{aligned} i[t/j] &= \begin{cases} i \text{ if } i < j \\ t \text{ if } i = j \\ i-1 \text{ if } i > j \end{cases} \\ (d_1\ d_2)[t/j] &= (d_1[t/j])\ (d_2[t/j]) \\ (\lambda\ d)[t/j] &= \lambda\ (d[t \uparrow_0 /j+1]) \end{aligned}$$

For the β-reduction, we only need to modify the case of the substition. In particular, we define $(\lambda\ d)\ e \to_\beta d[e/0]$.

Prove that $s[u/i] \to_\beta s'[u/i]$ if $s \to_\beta s'$.

Solution

Similarly to the fourth assertion of Lemma 1.1.5 in the lecture, we first prove the key property (*)

$$i < j+1 \longrightarrow t[v \uparrow_i /j+1][u[v/j]/i] = t[u/i][v/j]$$

by induction on t. Now

$$s \to_\beta s' \implies s[u/i] \to_\beta s'[u/i]$$

can be proved by induction on $\to_\beta$ for arbitrary u and i.

The base case is the hardest. We need to show

$$((\lambda\ s)\ t)[u/i] \to_\beta s[t/0][u/i]$$

for arbitrary s, t. Proof:

$$\begin{aligned}
& ((\lambda\ s)\ t)[u/i] \\
= \ & (\lambda\ s[u \uparrow_0 /i+1])\ t[u/i] && \text{Def. of substitution} \\
\to_\beta \ & s[u \uparrow_0 /i+1][t[u/i]/0] \\
= \ & s[t/0][u/i] && (*)
\end{aligned}$$

The other cases follow trivially from the rules of $\to_\beta$ and the definition of substitution.

Exercise 2 (Strong Confluence)

A relation $\to$ is said to be *strongly confluent* iff:

$$t_2 \leftarrow s \to t_1 \Longrightarrow \exists u.\ t_2 \to^= u \ {}^*\!\!\leftarrow t_1$$

Show that every *strongly confluent* relation is also *confluent*.

Solution

We show that every strongly confluent relation is also semi-confluent (see homework). To do so, we will show the stronger property

$$t_2 \ {}^n\!\!\leftarrow s \to t_1 \Longrightarrow \exists u.\ t_2 \to^= u \ {}^*\!\!\leftarrow t_1$$

by induction on n. The base case for $n = 0$ is easy: $s \ {}^0\!\!\leftarrow s \to t_1 \Longrightarrow s \to^= t_1 \ {}^*\!\!\leftarrow t_1$. For the induction step, we assume the statement for some u as the induction hypothesis. Furthermore, we assume $t_2' \leftarrow t_2 \ {}^n\!\!\leftarrow s \to t_1$ for some t_2'. We need to show that there exists a v with $t_2' \to^= v \ {}^*\!\!\leftarrow t_1$.

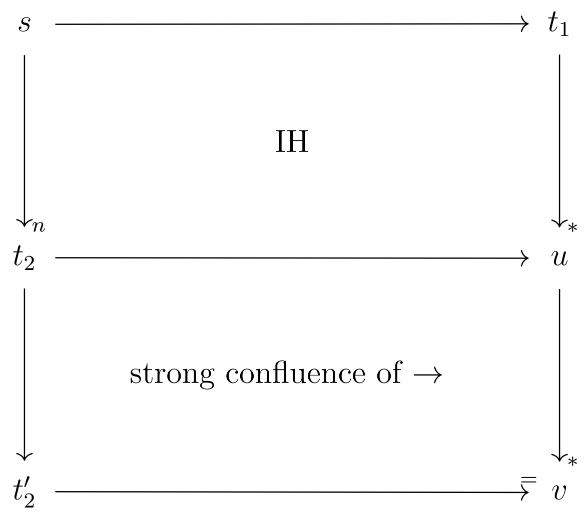
We make a case distinction on $t_2 \to^= u$.

When $t_2 = u$ we get $t_1 \to^* t_2$ with the IH and thus $t_2' \to^= t_2' \ {}^*\!\!\leftarrow t_1$ since $t_2 \to t_2'$.

If $t_2 \to u$, then from strong confluence of $\to$ with $t_2' \leftarrow t_2 \to u$ we obtain a v such that

$$u \to^* v \wedge t_2' \to^= v$$

Together with the induction hypothesis $t_1 \to^* u$, we get $t_2' \to^= v \; {}^*\!\!\leftarrow t_1$. As a picture:



Exercise 3 (Diamond Property & Normal Forms)

Show that if $\to$ has the diamond property, every element is either in normal form or has no normal form.

Solution

Assume that s is not in normal form but has a normal form t'. Then we have $t \to t'$ for some t. With the diamond property we get that there exists some t'' with $t' \to t''$. Thus t' is not a normal form, a contradiction.

Homework 4 (Semi-Confluence)

A relation $\rightarrow$ is said to be *semi-confluent* iff:

$$t_2 \ {}^*\!\!\leftarrow s \rightarrow t_1 \Longrightarrow \exists u.\ t_2 \rightarrow^* u \ {}^*\!\!\leftarrow t_1$$

Show that $\rightarrow$ is *semi-confluent* if and only if it is *confluent*.

Homework 5 (Weak Diamond Property)

Assume that $\rightarrow$ has the following weaker diamond property:

$$t_2 \leftarrow s \rightarrow t_1 \wedge t_1 \neq t_2 \Longrightarrow \exists u.\ t_2 \rightarrow u \leftarrow t_1.$$

a) Is it still the case that every element is either in normal form or has no normal form?

b) Show that if t has a normal form, then all its reductions to its normal form have the same length.