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Lambda Calculus
Winter Term 2023/24
Solutions to Exercise Sheet 11

Exercise 1 (Peirce's Law in Intuitionistic Logic)

Prove the following variant of Peirce's Law in inuitionistic logic:

$$((((P \to Q) \to P) \to P) \to Q) \to Q$$

Note: The original formulation of Peirce's law is $((P \to Q) \to P) \to P$. This implies the law of excluded middle, which is not provable in intuitionistic logic.

Solution

Let $A_3 = (P \to Q) \to P$, $A_2 = A_3 \to P$, and $A_1 = A_2 \to Q$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_1, A_3, P \vdash A_1 \qquad \dfrac{A_1, A_3, P \vdash P}{A_1, A_3, P \vdash A_2}\,{\to}\mathrm{I}}{A_1, A_3, P \vdash Q}\,{\to}\mathrm{E}}{A_1, A_3 \vdash P \to Q}\,{\to}\mathrm{I} \qquad A_1, A_3 \vdash A_3}{A_1, A_3 \vdash P}\,{\to}\mathrm{E}}{A_1 \vdash A_2}\,{\to}\mathrm{I} \qquad A_1 \vdash A_1}{A_1 \vdash Q}\,{\to}\mathrm{E}}{\vdash ((((P \to Q) \to P) \to P) \to Q) \to Q}\,{\to}\mathrm{I}
$$

Exercise 2 (Intuitionistic Proof Search in Haskell)

The goal of this exercise is to implement the procedure to decide $\Gamma \vdash A$ in Haskell, i.e. the algorithm from the proof of Theorem 4.1.2.

- Have a look at the template provided on the website. It provides definitions of formulae and proof terms of intuitionistic propositional logic.
- Try to fill in the implementation of *solve*.
- Implement the three proof rules seen in the lecture: *assumption*, *intro*, and *elim*. Use the examples at the end of the template to test your implementation as you go. For *elim*, use the criterion from the proof to guess suitable instantiations.

The algorithm can be streamlined further:

a) When trying to prove $\Gamma \vdash A \to B$, it suffices to try ($\to$Intro). Explain why.

b) The attempt to prove $\Gamma \vdash A$ by assumption can be dropped if we use the following generalised →Elim rule:

$$\frac{\Gamma \vdash A_1 \to \ldots \to A_n \to B \qquad \forall i \leq n.\ \ \Gamma \vdash A_i}{\Gamma \vdash B} \to\textsc{Elim}$$

However, the proof obtained can be different. Explain the difference and why the outright proof by assumption is subsumed.

Solution

See `prover_sol.hs` for the implementation.

In the following we will denote the by (→Elim) the more general rule described in lemma 4.1.2.

a) Suppose we prove $\Gamma \vdash A \to B$ by an application of (→Elim). The proof will be of the following format:

$$\frac{\Gamma \vdash A_1 \to \ldots \to A_n \to A \to B \qquad \forall i \leq n.\ \ \Gamma \vdash A_i}{\Gamma \vdash A \to B} \to\textsc{Elim}$$

We can always provide an alternative proof that uses (→Intro) first and looks like this:

$$\frac{\dfrac{\Gamma \vdash A_1 \to \ldots \to A_n \to A \to B \qquad \forall i \leq n.\ \ \Gamma, A \vdash A_i \qquad \Gamma, A \vdash A}{\Gamma, A \vdash B} \to\textsc{Elim}}{\Gamma \vdash A \to B} \to\textsc{Intro}$$

The case where $\Gamma \vdash A \to B$ is proved by assumption is subsumed by the next answer.

b) Proof by assumption is just a special case of (→Elim) where $n = 0$. However, if we drop the assumption rule, proofs can now have a slightly different structure because we try (→Intro) first:

$$\frac{\dfrac{A_1 \to \ldots \to A_n \to B \in \Gamma' \qquad \forall i \leq n.\ \ \Gamma' \vdash A_i}{\Gamma' \vdash B} \to\textsc{Elim}}{\Gamma, A_1 \to \ldots \to A_n \to B \vdash A_1 \to \ldots \to A_n \to B} \to\textsc{Intro}\ n\ \textsc{times}$$

with

$$\Gamma' := \Gamma, A_1 \to \ldots \to A_n \to B, A_1, \ldots, A_n\,.$$

Homework 3 (Constructive Logic)

a) Prove the following statement using the calculus for intuitionistic propositional logic:

$$((c \to b) \to b) \to (c \to a) \to ((a \to b) \to b)$$

Hint: To make your proof tree more compact, you may remove unneeded assumptions to the left of the $\vdash$ during the proof as you see fit. For example, the following step is valid:

$$\frac{p \vdash p}{p, q \vdash p}$$

b) Give a well-typed expression in $\lambda^{\to}$ with the type

$$((\gamma \to \beta) \to \beta) \to (\gamma \to \alpha) \to ((\alpha \to \beta) \to \beta)$$

(You don't need to give the derivation tree.)

Homework 4 (The Negative Fragment)

We say that a formula A is negative if atomic formulas P only occur *negated* in A, i.e. in the form $P \to \bot$ ($\neg P$ for short). The symbol $\bot$ for *falsehood* plays the role of an unprovable propositional constant: we do not have any special proof rules or axioms for it.

Show that if A is negative, then:

$$\vdash \neg\neg A \to A$$

Hint: First show:

a) $\vdash \neg\neg\neg A \to \neg A$

b) $\vdash \neg\neg(A \to B) \to (\neg\neg A \to \neg\neg B)$

c) $\vdash (\neg\neg A \to \neg\neg B) \to (A \to \neg\neg B)$