# Propositional Logic
# Compactness

# Compactness Theorem

Theorem

*A set $S$ of formulas is satisfiable*
*iff every finite subset of $S$ is satisfiable.*

Equivalent formulation:

A set $S$ of formulas is unsatisfiable
iff some finite subset of $S$ is unsatisfiable.

# An application: Graph Coloring

### Definition

A 4-coloring of a graph $(V, E)$ is a map $c : V \to \{1, 2, 3, 4\}$ such that $(x, y) \in E$ implies $c(x) \neq c(y)$.

### Theorem

*An finite planar graph has a 4-coloring.*

### Theorem (4CT)

*An planar graph* $G = (V, E)$ *with countably many vertices* $V = \{v_1, v_2, \ldots\}$ *has a 4-coloring.*

**Proof** $G \leadsto$ set of formulas $S$ s.t. $S$ is sat. iff $G$ is 4-col.
$G$ is planar
$\Rightarrow$ every finite subgraph of $G$ is planar and 4-col. (by 4CT)
$\Rightarrow$ every finite subset of $S$ is sat.
$\Rightarrow$ $S$ is sat. (by Compactness)
$\Rightarrow$ $G$ is 4-col.

# Proof details

$G \rightsquigarrow S$:

For simplicity:
atoms are of the form $A_i^c$ where $c \in \{1, \ldots, 4\}$ and $i \in \mathbb{N}$

$$\begin{aligned} S := \quad & \{A_i^1 \vee A_i^2 \vee A_i^3 \vee A_i^4 \mid i \in \mathbb{N}\} \cup \\ & \{A_i^c \to \neg A_i^d \mid i \in \mathbb{N},\ c, d \in \{1, \ldots, 4\},\ c \neq d\} \cup \\ & \{\neg(A_i^c \wedge A_j^c) \mid (v_i, v_j) \in E,\ c \in \{1, \ldots, 4\}\} \end{aligned}$$

Subgraph corresponding to some $T \subseteq S$:
$V_T := \{v_i \mid A_i^c \text{ occurs in } T \text{ (for some } c)\}$
$E_T := \{(v_i, v_j) \mid \neg(A_i^c \wedge A_j^c) \in T \text{ (for some } c)\}$

# Proof of Compactness

Theorem
*A set $S$ of formulas is satisfiable*
*iff every finite subset of $S$ is satisfiable.*

**Proof**

$\Rightarrow$: If $S$ is satisfiable then every finite subset of $S$ is satisfiable.

Trivial.

$\Leftarrow$ : If every finite subset of $S$ is satisfiable then $S$ is satisfiable.

We prove that $S$ has a model.

# Proof of Compactness

Terminology: $\mathcal{A}$ is a $b_1, \ldots, b_n$ model of $T$
(where $b_1, \ldots, b_n \in \{0,1\}^*$ and $T$ is a set of formulas)
if $\mathcal{A}(A_i) = b_i$ (for $i = 1, \ldots, n$) and $\mathcal{A} \models T$.

Define an infinite sequence $b_1, b_2, \ldots$ recursively as follows:

$$b_{n+1} = \text{some } b \in \{0,1\} \text{ s.t. all finite } T \subseteq S \text{ have a } b_1, \ldots, b_n, b \text{ model.}$$

**Claim 1:** For all $n$, all finite $T \subseteq S$ have a $b_1, \ldots, b_n$ model.
**Proof** by induction on $n$.

Case $n = 0$: because all finite $T \subseteq S$ are satisfiable.

Case $n + 1$: We need to show that a suitable $b$ exists.
Proof by contradiction. Assume there is no suitable $b$.
Then there is a finite $T_0 \subseteq S$ that has no $b_1, \ldots, b_n, 0$ model (0)
and there is a finite $T_1 \subseteq S$ that has no $b_1, \ldots, b_n, 1$ model (1).
Therefore $T_0 \cup T_1$ has no $b_1, \ldots, b_n$ model $\mathcal{A}$:
$\mathcal{A}(A_{n+1}) = 0$ contradicts (0), $\mathcal{A}(A_{n+1}) = 1$ contradicts (1).
But by IH: $T_0 \cup T_1$ has a $b_1, \ldots, b_n$ model — Contradiction!

# Proof of Compactness

Define $\mathcal{B}(A_i) = b_i$ for all $i$.

**Claim 2:** $\mathcal{B} \models S$
We show $\mathcal{B} \models F$ for all $F \in S$.
Let $m$ be the maximal index of all atoms in $F$.
By Claim 1, $\{F\}$ has a $b_1, \ldots, b_m$ model $\mathcal{A}$.
Hence $\mathcal{B} \models F$ because $\mathcal{A}$ and $\mathcal{B}$ agree on all atoms in $F$.

# Corollary

**Corollary**
*If $S \models F$ then there is a finite subset $M \subseteq S$ such that $M \models F$.*