Propositional Logic
Equivalences

Equivalence

Definition (Equivalence)

Two formulas F and G are (semantically) equivalent if $\mathcal{A}(F) = \mathcal{A}(G)$ for every assignment $\mathcal{A}$.

We write $F \equiv G$ to denote that F and G are equivalent.

Exercise

Which of the following equivalences hold?

$$
\begin{aligned}
(A \wedge (A \vee B)) &\equiv A \\
(A \wedge (B \vee C)) &\equiv ((A \wedge B) \vee C) \\
(A \rightarrow (B \rightarrow C)) &\equiv ((A \rightarrow B) \rightarrow C) \\
(A \rightarrow (B \rightarrow C)) &\equiv ((A \wedge B) \rightarrow C)
\end{aligned}
$$

Observation

The following connections hold:

$$\models F \rightarrow G \quad \text{iff} \quad F \models G$$
$$\models F \leftrightarrow G \quad \text{iff} \quad F \equiv G$$

NB: “iff” means “if and only if”

Reductions between problems (I)

- Validity to Unsatisfiabilty (and back):

$$F \text{ valid} \quad \text{iff} \quad \neg F \text{ unsatisfiable}$$
$$F \text{ unsatisfiable} \quad \text{iff} \quad \neg F \text{ valid}$$

- Validity to Consequence:

$$F \text{ valid} \quad \text{iff} \quad \top \models F$$

- Consequence to Validity:

$$F \models G \quad \text{iff} \quad F \rightarrow G \text{ valid}$$

Reductions between problems (II)

- Validity to Equivalence:

$$F \text{ valid} \quad \text{iff} \quad F \equiv \top$$

- Equivalence to Validity:

$$F \equiv G \quad \text{iff} \quad F \leftrightarrow G \text{ valid}$$

Properties of semantic equivalence

- Semantic equivalence is an equivalence relation between formulas.
- Semantic equivalence is closed under operators:

  If $F_1 \equiv F_2$ and $G_1 \equiv G_2$
  then $(F_1 \wedge G_1) \equiv (F_2 \wedge G_2)$,
  $(F_1 \vee G_1) \equiv (F_2 \vee G_2)$ and
  $\neg F_1 \equiv \neg F_2$

Equivalence relation + Closure under Operations

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Congruence relation

Replacement theorem

Theorem

Let $F \equiv G$. Let H be a formula with an occurrence of F as a subformula. Let H' be the result of replacing an arbitrary occurrence of F in H by G. Then $H \equiv H'$.

Proof by induction on the structure of H.
We consider only the case $H = \neg H_0$.
We analyse where F occurs in H.
If $F = H$ then $H' = G$ and thus $H = F \equiv G = H'$.
Otherwise F is a subformula of H_0.
Let H_0' be the result of replacing F by G in H_0.
IH: $H_0 \equiv H_0'$
Thus $H = \neg H_0 \equiv \neg H_0' = H'$

Equivalences (I)

Theorem

$$
\begin{array}{rcll}
(F \wedge F) & \equiv & F & \\
(F \vee F) & \equiv & F & \textit{(Idempotence)} \\
(F \wedge G) & \equiv & (G \wedge F) & \\
(F \vee G) & \equiv & (G \vee F) & \textit{(Commutativity)} \\
((F \wedge G) \wedge H) & \equiv & (F \wedge (G \wedge H)) & \\
((F \vee G) \vee H) & \equiv & (F \vee (G \vee H)) & \textit{(Associativity)} \\
(F \wedge (F \vee G)) & \equiv & F & \\
(F \vee (F \wedge G)) & \equiv & F & \textit{(Absorption)}
\end{array}
$$

Equivalences (II)

$$
\begin{array}{rcll}
(F \wedge (G \vee H)) & \equiv & ((F \wedge G) \vee (F \wedge H)) & \\
(F \vee (G \wedge H)) & \equiv & ((F \vee G) \wedge (F \vee H)) & \text{(Distributivity)} \\
\neg\neg F & \equiv & F & \text{(Double negation)} \\
\neg(F \wedge G) & \equiv & (\neg F \vee \neg G) & \\
\neg(F \vee G) & \equiv & (\neg F \wedge \neg G) & \text{(deMorgan's Laws)} \\
\neg\top & \equiv & \bot & \\
\neg\bot & \equiv & \top & \\
(\top \vee G) & \equiv & \top & \\
(\top \wedge G) & \equiv & G & \\
(\bot \vee G) & \equiv & G & \\
(\bot \wedge G) & \equiv & \bot &
\end{array}
$$

Warning

The symbols $\models$ and $\equiv$ are not operators
in the language of propositional logic
but part of the meta-language for talking about logic.

Examples:

$\mathcal{A} \models F$ and $F \equiv G$ are not propositional formulas.

$(\mathcal{A} \models F) \equiv G$ and $(F \equiv G) \leftrightarrow (G \equiv F)$ are nonsense.