Hilbert Systems Propositional Logic

(See the book by Troelstra and Schwichtenberg)

Easy to define, hard to use No context management

A Hilber system for propositional logic consists of

- a set of axioms (formulae)
- ▶ and a single infrence rule, $\rightarrow E$ or modus ponens:

$$\frac{F \to G \quad F}{G} \to E$$

Proof trees for some Hilbert system are labeled with formulas. The only inference rule is $\rightarrow E$.

Definition

We write $\Gamma \vdash_H F$ if there is a proof tree with root F whose leaves are either axioms or elements of Γ .

Alternative proof presentation

Proofs in Hilbert systems are freqently shown as lists of lines

- 1. F₁ justification₁
- 2. F₂ justification₂
- i. F_i justification_i

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justification<sub>i</sub> is either
assumption, axiom or \rightarrow E(j, k) where j, k < i
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Like linearized tree but also allows sharing of subproofs

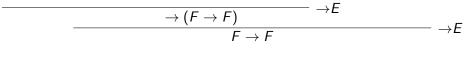
Notational convention:

$$F \to G \to H \quad \text{means} \quad F \to (G \to H)$$

Note:
$$F \to G \to H \equiv F \land G \to H$$
$$F \to G \to H \not\equiv (F \to G) \to H$$

Example (A simple Hilbert system)Axioms: $F \rightarrow (G \rightarrow F)$ (A1) $(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$ (A2)

A proof of $F \rightarrow F$:



 $\Rightarrow \vdash_H F \rightarrow F$

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

 $F, \Gamma \vdash_H G \quad iff \quad \Gamma \vdash_H F \to G$

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Proof "\Leftarrow":

\Gamma \vdash_H F \to G

\Rightarrow F, \Gamma \vdash_H F \to G

\Rightarrow F, \Gamma \vdash_H G by \to E because F, \Gamma \vdash_H F
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Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

 $F, \Gamma \vdash_H G \quad iff \quad \Gamma \vdash_H F \to G$

Proof " \Rightarrow ": By induction on (the length/depth of) the proof of $F, \Gamma \vdash_H G$ Then by cases on the last proof step:

Case G = F: see proof of $F \to F$ from A1 and A2 Case $G \in \Gamma$ or axiom: by A1 and ...

Case $\rightarrow E$ from $H \rightarrow G$ and H:

$$\frac{(F \to H \to G) \to (F \to H) \to F \to G \quad F \to H \to G}{(F \to H) \to F \to G} \quad F \to H}{F \to G}$$

Hilbert System

From now on \vdash_{H} refers to the following set of axioms:

$$\begin{array}{ll} F \rightarrow G \rightarrow F & (A1) \\ (F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H & (A2) \\ F \rightarrow G \rightarrow F \wedge G & (A3) \\ F \wedge G \rightarrow F & (A4) \\ F \wedge G \rightarrow G & (A5) \\ F \rightarrow F \lor G & (A5) \\ G \rightarrow F \lor G & (A7) \\ F \lor G \rightarrow (F \rightarrow H) \rightarrow (G \rightarrow H) \rightarrow H & (A8) \\ (\neg F \rightarrow \bot) \rightarrow F & (A9) \end{array}$$

Relating Hilbert and Natural Deduction

Theorem (Hilbert can simulate ND)

If $\Gamma \vdash_N F$ then $\Gamma \vdash_H F$

Proof translation in two steps: $\vdash_N \quad \rightsquigarrow \quad \vdash_H \quad + \rightarrow I \quad \rightsquigarrow \quad \vdash_H$

- Transform a ND-proof tree into a proof tree containing Hilbert axioms, →E and →I by replacing all other ND rules by Hilbert proofs incl. →I Principle: ND rule → 1 axiom + →I/E
- 2. Eliminate the \rightarrow *I* rules by the Deduction Theorem

Lemma (ND can simulate Hilbert) If $\Gamma \vdash_H F$ then $\Gamma \vdash_N F$ **Proof** by induction on $\Gamma \vdash_H F$.

- Every Hilbert axiom is provable in ND (Exercise!)
- $\blacktriangleright \rightarrow E$ is also available in ND

 $\begin{array}{l} \mathsf{Corollary} \\ \mathsf{\Gamma}\vdash_{H}\mathsf{F} \quad iff \ \mathsf{\Gamma}\vdash_{N}\mathsf{F} \end{array}$

Corollary (Soundness and completeness) $\Gamma \vdash_H F \quad iff \quad \Gamma \models F$