Propositional Logic
Resolution

Clause representation of CNF formulas

CNF:

$$(L_{1,1} \vee \ldots \vee L_{1,n_1}) \wedge \ldots \wedge (L_{k,1} \vee \ldots \vee L_{1,n_k})$$

Representation as set of sets of literals:

$$\{\underbrace{\{L_{1,1},\ \ldots,\ L_{1,n_1}\}}_{\textit{clause}},\ \ldots,\ \{L_{k,1},\ \ldots,\ L_{1,n_k}\}\}$$

- Clause $=$ set of literals (disjunction).
- A formula in CNF can be viewed as a set of clauses
- Degenerate cases:
  - The empty clause stands for $\perp$.
  - The empty set of clauses stands for $\top$.

The joy of sets

We get "for free":

- Commutativity:
  $A \vee B \equiv B \vee A$, both represented by $\{A, B\}$
- Associativity:
  $(A \vee B) \vee C \equiv A \vee (B \vee C)$, both represented by $\{A, B, C\}$
- Idempotence:
  $(A \vee A) \equiv A$, both represented by $\{A\}$

Sets are a convenient representation of conjunctions and disjunctions that build in associativity, commutativity and itempotence

Resolution — The idea

Input: Set of clauses F
Question: Is F unsatisfiable?

Algorithm:
Keep on "resolving" two clauses from F and adding the result to F until the empty clause is found

Correctness:
If the empty clause is found, the initial F is unsatisfiable
Completeness:
If the initial F is unsatisfiable, the empty clause can be found.

Correctness/Completeness of syntactic procedure (resolution) w.r.t. semantic property (unsatisfiability)

Resolvent

Definition

Let L be a literal. Then $\overline{L}$ is defined as follows:

$$\overline{L} = \begin{cases} \neg A_i & \text{if } L = A_i \\ A_i & \text{if } L = \neg A_i \end{cases}$$

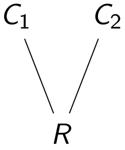
Definition

Let C_1, C_2 be clauses and let L be a literal
such that $L \in C_1$ and $\overline{L} \in C_2$. Then the clause

$$(C_1 - \{L\}) \cup (C_2 - \{\overline{L}\})$$

is a resolvent of C_1 and C_2.
The process of deriving the resolvent is called a resolution step.

Graphical representation of resolvent:



If $C_1 = \{L\}$ and $C_2 = \{\overline{L}\}$ then the empty clause is a resolvent of C_1 and C_2. The special symbol $\Box$ denotes the empty clause.

Recall: $\Box$ represents $\bot$.

Resolution proof

Definition

A resolution proof of a clause C from a set of clauses F
is a sequence of clauses $C_0, \ldots, C_n$ such that

- $C_i \in F$ or C_i is a resolvent of two clauses C_a and C_b, $a, b < i$,
- $C_n = C$

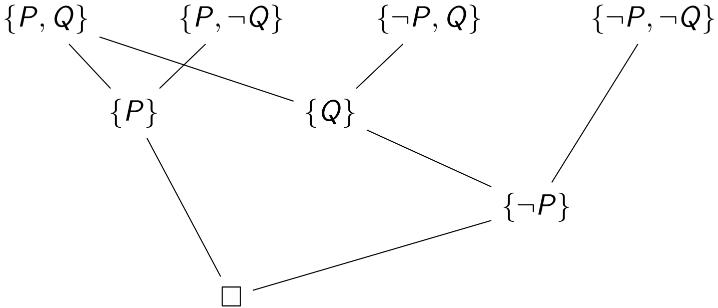
Then we can write $F \vdash_{Res} C$.

Note: F can be finite or infinite

Resolution proof as DAG

A resolution proof can be shown as a DAG with the clauses in F as the leaves and C as the root:

Example



A linear resolution proof

0:	$\{P, Q\}$	
1:	$\{P, \neg Q\}$	
2:	$\{\neg P, Q\}$	
3:	$\{\neg P, \neg Q\}$	
4:	$\{P\}$	(0, 1)
5:	$\{Q\}$	(0, 2)
6:	$\{\neg P\}$	(3, 5)
7:	$\square$	(4, 6)

Correctness of resolution

Lemma (Resolution Lemma)

Let R be a resolvent of two clauses C_1 and C_2. Then $C_1, C_2 \models R$.

Proof By definition $R = (C_1 - \{L\}) \cup (C_2 - \{\bar{L}\})$ (for some L).
Let $\mathcal{A} \models C_1$ and $\mathcal{A} \models C_2$. There are two cases.
If $\mathcal{A} \models L$ then $\mathcal{A} \models C_2 - \{\bar{L}\}$ (because $\mathcal{A} \models C_2$), thus $\mathcal{A} \models R$.
If $\mathcal{A} \not\models L$ then $\mathcal{A} \models C_1 - \{L\}$ (because $\mathcal{A} \models C_1$), thus $\mathcal{A} \models R$.

Theorem (Correctness of resolution)

Let F be a set of clauses. If $F \vdash_{Res} C$ then $F \models C$.

Proof Assume there is a resolution proof $C_0, \ldots, C_n = C$.
By induction on i we show $F \models C_i$. IH: $F \models C_j$ for all $j < i$.
If $C_i \in F$ then $F \models C_i$ is trivial. If C_i is a resolvent of C_a and C_b, $a, b < i$, then $F \models C_a$ and $F \models C_b$ by IH and $C_a, C_b \models C_i$ by the resolution lemma. Thus $F \models C_i$.

Corollary

Let F be a set of clauses. If $F \vdash_{Res} \Box$ then F is unsatisfiable.

Completeness of resolution

Theorem

Let F be a finite set of clauses. If F is unsatisfiable then $F \vdash_{Res} \square$.

Theorem (Completeness of resolution)

Let F be a set of clauses. If F is unsatisfiable then $F \vdash_{Res} \square$.

Proof If F is infinite, there must be a finite unsatisfiable subset of F (by the Compactness Theorem); in that case let F be that finite subset and apply the previous theorem.

Corollary

A set of clauses F is unsatisfiable iff $F \vdash_{Res} \square$.

Completeness proof

Theorem

Let F be a finite set of clauses. If F is unsatisfiable then $F \vdash_{Res} \Box$.

Proof The proof of $F \vdash_{Res} \Box$ is by induction on the number n of distinct atoms in F.

Basis: If $n = 0$ then $F = \{\}$ (but F is unsat.) or $F = \{\Box\}$.

Step:

IH: For every unsat. set of clauses F with n dist. atoms, $F \vdash_{Res} \Box$.
Let F contain $n + 1$ distinct atoms. Pick some atom A in F.
Idea: $F_0 = F$ with A replaced by $\bot$
$F_1 := F$ with A replaced by $\top$
$F_0 :=$ take F, remove all clauses with $\neg A$, remove all A
$F_1 :=$ take F, remove all clauses with A, remove all $\neg A$
F_0 and F_1 contain n distinct atoms.
F_0 is unsat: if $\mathcal{A} \models F_0$ then $\mathcal{A}[0/A] \models F$
F_1 is unsat: if $\mathcal{A} \models F_1$ then $\mathcal{A}[1/A] \models F$

Completeness proof

By IH: there are res. proofs $C_0, \dots, C_m = \square$ from F_0 and $D_0, \dots, D_n = \square$ from F_1.
Now transform $C_0, \dots, C_m$ into a proof $C'_0, \dots, C'_m$ from F by adding A back into the clauses it was removed from. Then

- either $C'_m = \{A\}$
- or $C'_m = \square$ (and we are done).

Similarly we transform $D_0, \dots, D_n$ into a proof $D'_0, \dots, D'_n$ from F (by adding $\neg A$ back in).
Then $D'_n = \{\neg A\}$ or $D'_n = \square$ (and we are done).
If $C'_m = \{A\}$ and $D'_n = \{\neg A\}$ then $F \vdash_{Res} A$ and $F \vdash_{Res} \neg A$ and thus $F \vdash_{Res} \square$.

Resolution is only refutation complete

Not everything that is a consequence of a set of clauses
can be derived by resolution.

Exercise
Find F and C such that $F \models C$ *but not* $F \vdash_{Res} C$.

How to prove $F \models C$ by resolution?
Prove $F \cup \{\neg C\} \vdash_{Res} \square$

A resolution algorithm

Input: A CNF formula F, i.e. a finite set of clauses

while there are clauses $C_a, C_b \in F$ and resolvent R of C_a and C_b
such that $R \notin F$
do $F := F \cup \{R\}$

Lemma

The algorithm terminates.

Proof There are only finitely many clauses over a finite set of atoms.

Theorem

The initial F is unsatisfiable iff $\Box$ is in the final F

Proof F_{init} is unsat. iff $F_{init} \vdash_{Res} \Box$ iff $\Box \in F_{final}$
because the algorithm enumerates all R such that $F_{init} \vdash R$.

Corollary

The algorithm is a decision procedure for unsatisfiability of CNF formulas.