First-Order Logic
Equality

Predicate logic with equality

Predicate logic

$+$

distinguished predicate symbol "=" of arity 2

Semantics: A structure $\mathcal{A}$ of predicate logic with equality always maps the predicate symbol = to the identity relation:

$$\mathcal{A}(=) \; = \; \{(d, d) \mid d \in U_{\mathcal{A}}\}$$

Expressivity

Fact

A structure is model of $\exists x \forall y\ x{=}y$ iff its universe is a singleton.

Theorem

Every satisfiable formula of predicate logic has a countably infinite model.

Proof Let F be satisfiable. We assume w.l.o.g. that $F = \forall x_1 \dots \forall x_n F^*$ and the variables occurring in F^* are exactly $x_1, \dots, x_n$. (If necessary bring F into closed Skolem form).
We consider two cases:
$n = 0$. Exercise.
$n > 0$. Let $G = \forall x_1 \dots \forall x_n F^*[f(x_1)/x_1]$, where f is a function symbol that does not occur in F^*. G is satisfiable (why?).
If G has a model M with universe U, then F has a model with universe $\{f^M(u) \mid u \in (U)\}$. Because G has a Herbrand model with countably infinite universe $T(G)$ (by the Fundamental Theorem), F also has a model with countably infinite universe $\{f(t) \mid t \in T(G)\}$.

Modelling equality

Let F be a formula of predicate logic with equality.
Let Eq be a predicate symbol that does not occur in F.
Let E_F be the conjunction of the following formulas:

$$\forall x\ Eq(x, x)$$

$$\forall x \forall y\ (Eq(x, y) \rightarrow Eq(y, x))$$

$$\forall x \forall y \forall z\ ((Eq(x, y) \wedge Eq(y, z)) \rightarrow Eq(x, z))$$

For every function symbol f in F of arity n and every $1 \leq i \leq n$:

$$\forall x_1 \ldots \forall x_n \forall y\ (Eq(x_i, y) \rightarrow Eq(f(x_1, \ldots, x_i, \ldots x_n), f(x_1, \ldots, y, \ldots, x_n)))$$

For every predicate symbol P in F of arity n and every $1 \leq i \leq n$:

$$\forall x_1 \ldots \forall x_n \forall y (Eq(x_i, y) \rightarrow (P(x_1, \ldots, x_i, \ldots, x_n) \leftrightarrow P(x_1, \ldots, y, \ldots, x_n)))$$

E_F expresses that Eq is a *congruence relation* on the symbols in F.

Quotient structure

Definition

Let $\mathcal{A}$ be a structure and $\sim$ an equivalence relation on $U_{\mathcal{A}}$ that is a congruence relation for all the predicate and function symbols defined by $I_{\mathcal{A}}$. The quotient structure $\mathcal{A}/_{\sim}$ is defined as follows:

- $U_{\mathcal{A}/_{\sim}} = \{[u]_{\sim} \mid u \in U_{\mathcal{A}}\}$ where $[u]_{\sim} = \{v \in U_{\mathcal{A}} \mid u \sim v\}$
- For every function symbol f defined by $I_{\mathcal{A}}$: $f^{\mathcal{A}/\sim}([d_1]_{\sim}, \ldots, [d_n]_{\sim}) = [f^{\mathcal{A}}(d_1, \ldots, d_n)]_{\sim}$
- For every predicate symbol P defined by $I_{\mathcal{A}}$: $P^{\mathcal{A}/\sim}([d_1]_{\sim}, \ldots, [d_n]_{\sim}) = P^{\mathcal{A}}(d_1, \ldots, d_n)$
- For every variable x defined by $I_{\mathcal{A}}$: $x^{\mathcal{A}/\sim} = [x^{\mathcal{A}}]_{\sim}$

Lemma

$\mathcal{A}/_{\sim}(t) = [\mathcal{A}(t)]_{\sim}$

Lemma

$\mathcal{A}/_{\sim}(F) = \mathcal{A}(F)$

Theorem

The formulas F and $E_F \wedge F[Eq/{=}]$ are equisatisfiable.

Proof We show that if $E_F \wedge F[Eq/{=}]$ is sat., then F is satisfiable.
Assume $\mathcal{A} \models E_F \wedge F[Eq/{=}]$.
$\Rightarrow$ $Eq^{\mathcal{A}}$ is an congruence relation.

Let $\mathcal{B} = \mathcal{A}/_{Eq^{\mathcal{A}}}$ (extended with $=$ interpreted as identity).
$\Rightarrow \mathcal{B} \models F[Eq/{=}]$

By construction $Eq^{\mathcal{B}}$ is identity:
$Eq^{\mathcal{B}}([a],[a']) = Eq^{\mathcal{A}}(a,a') = ([a]_{Eq^{\mathcal{A}}} = [a']_{Eq^{\mathcal{A}}})$
$\Rightarrow \mathcal{B}(F[Eq/{=}]) = \mathcal{B}(F)$
$\Rightarrow \mathcal{B} \models F$

Conversely, it is easy to see that any model of F can be turned into a model of $E_F \wedge F[Eq/{=}]$ by interpreting Eq as equality.

Higher-Order Logic (HOL)

Types and Terms

Simly typed λ-terms

Types:

$$\begin{array}{lll} \tau & ::= & bool \mid \ldots \\ & \mid & (\tau \rightarrow \tau) \\ & \mid & \alpha \mid \beta \ldots \end{array}$$

Terms

$$\begin{array}{lll} t & ::= & c \mid d \mid \cdots \mid f \mid h \mid \ldots \\ & \mid & (t\ t) \\ & \mid & (\lambda x.\ t) \end{array}$$

We assume that every variable and constant has an attached type.
We consider only well-typed terms:

$$\frac{t_1 : \tau \rightarrow \tau' \quad t_2 : \tau}{t_1\ t_2 : \tau'} \qquad \frac{t : \tau'}{\lambda x{:}\tau.\ t : \tau \rightarrow \tau'}$$

Base logic

Formula $=$ term of type *bool*

Theorems: $\Gamma \vdash F$

Base constants: $= : \alpha \to \alpha \to bool$
$\to : bool \to bool \to bool$

Inference rules

$$\frac{}{F \vdash F}\ \textit{assume}$$

$$\frac{}{\vdash t = t}\ \textit{refl}$$

$$\frac{}{\vdash (\lambda x.\ t)\ u = u[t/x]}\ \beta$$

$$\frac{}{\vdash \lambda x.\ (t\ x) = t}\ \eta \quad \text{if } x \notin \mathit{fv}(t)$$

$$\frac{\Gamma_1 \vdash s = t \quad \Gamma_2 \vdash F[s/x]}{\Gamma_1 \cup \Gamma_2 \vdash F[t/x]}\ \textit{subst}$$

$$\frac{\Gamma \vdash s = t}{\Gamma \vdash (\lambda x.\ s) = (\lambda x.\ t)}\ \textit{abs} \quad \text{if } x \notin \mathit{fv}(\Gamma)$$

Inference rules

$$\frac{\Gamma \vdash F}{\Gamma \vdash F[\tau_1/\alpha_1, \ldots]}\ \textit{inst}$$

if $\alpha_1, \ldots$ do not occur in Γ

Inference rules

$$\frac{\Gamma \vdash G}{\Gamma \backslash \{F\} \vdash F \rightarrow G} \rightarrow I$$

$$\frac{\Gamma_1 \vdash F \rightarrow G \quad \Gamma_2 \vdash F}{\Gamma_1 \cup \Gamma_2 \vdash G} \rightarrow E$$

$$\frac{\Gamma_1 \vdash F \rightarrow G \quad \Gamma_2 \vdash G \rightarrow F}{\Gamma_1 \cup \Gamma_2 \vdash F = G} =I$$

Definitions of standard logical symbols

$$\vdash \top = ((\lambda x.\ x) = (\lambda x.\ x))$$

$all : (\alpha \to bool) \to bool$
Notation: $\forall x.\ F$ abbreviates $all(\lambda x.\ F)$

$$\vdash all = (\lambda P.\ P = (\lambda x.\ \top))$$

$$\vdash \bot = (\forall F.\ F)$$

$$\vdash \neg = (\lambda F.\ F \to \bot)$$

$$\vdash (\wedge) = (\lambda F.\ \lambda G.\ \forall H.\ (F \to G \to H) \to H)$$

$$\vdash (\vee) = (\lambda F.\ \lambda G.\ \forall H.\ (F \to H) \to (G \to H) \to H)$$

Definitions of standard logical symbols

$ex : (\alpha \to bool) \to bool$
Notation: $\exists x.\ F$ abbreviates $ex(\lambda x.\ F)$

$$\vdash ex = (\lambda P.\ \forall G.\ (\forall x.\ (P\ x \to G) \to G))$$

> *The method of postulating what we want has many advantages; they are the same as the advantages of theft over honest toil.*

Bertrand Russel

Classical logic

$$\vdash F \lor \neg F$$

Hilbert's ε

Informally: $\varepsilon\, x.\, F$ = an arbitrary but fixed x that satisfies F

Examples

$(\varepsilon\, x.\, x = 5) = 5$
$(\varepsilon\, n.\, 0 \leq n \leq 2) \in \{0, 1, 2\}$
$(\varepsilon\, x.\, \bot)$???

Formally: $eps : (\alpha \to bool) \to \alpha$
$\varepsilon\, x.\, F$ appreviates $eps(\lambda x.\, F)$
Axiom: $P\ x \to P(eps\ P)$