Higher-Order Logic (HOL)

Types and Terms

Simly typed λ-terms

Types:

$$
\begin{array}{lll}
\tau & ::= & \mathit{bool} \mid \ldots \\
 & \mid & (\tau \to \tau) \\
 & \mid & \alpha \mid \beta \ldots
\end{array}
$$

Terms

$$
\begin{array}{lll}
t & ::= & c \mid d \mid \cdots \mid f \mid h \mid \ldots \\
 & \mid & (t\ \ t) \\
 & \mid & (\lambda x.\ t)
\end{array}
$$

We assume that every variable and constant has an attached type.
We consider only well-typed terms:

$$
\frac{t_1 : \tau \to \tau' \quad t_2 : \tau}{t_1\ \ t_2 : \tau'} \qquad \frac{t : \tau'}{\lambda x{:}\tau.\ t : \tau \to \tau'}
$$

Base logic

Formula $=$ term of type *bool*

Theorems: $\Gamma \vdash F$

Base constants: $= : \alpha \to \alpha \to$ *bool*
$\to$: *bool* $\to$ *bool* $\to$ *bool*

Inference rules

$$\frac{}{F \vdash F}\ \textit{assume}$$

$$\frac{}{\vdash t = t}\ \textit{refl}$$

$$\frac{}{\vdash (\lambda x.\ t)\ u = u[t/x]}\ \beta$$

$$\frac{}{\vdash \lambda x.\ (t\ x) = t}\ \eta \quad \text{if } x \notin \mathit{fv}(t)$$

$$\frac{\Gamma_1 \vdash s = t \quad \Gamma_2 \vdash F[s/x]}{\Gamma_1 \cup \Gamma_2 \vdash F[t/x]}\ \textit{subst}$$

$$\frac{\Gamma \vdash s = t}{\Gamma \vdash (\lambda x.\ s) = (\lambda x.\ t)}\ \textit{abs} \quad \text{if } x \notin \mathit{fv}(\Gamma)$$

Inference rules

$$\frac{\Gamma \vdash F}{\Gamma \vdash F[\tau_1/\alpha_1, \ldots]} \; \textit{inst}$$

if $\alpha_1, \ldots$ do not occur in Γ

Inference rules

$$\frac{\Gamma \vdash G}{\Gamma \backslash \{F\} \vdash F \to G} \to I$$

$$\frac{\Gamma_1 \vdash F \to G \quad \Gamma_2 \vdash F}{\Gamma_1 \cup \Gamma_2 \vdash G} \to E$$

$$\frac{\Gamma_1 \vdash F \to G \quad \Gamma_2 \vdash G \to F}{\Gamma_1 \cup \Gamma_2 \vdash F = G} =I$$

Definitions of standard logical symbols

$$\vdash \top = ((\lambda x.\ x) = (\lambda x.\ x))$$

$all : (\alpha \to bool) \to bool$
Notation: $\forall x.\ F$ abbreviates $all(\lambda x.\ F)$

$$\vdash all = (\lambda P.\ P = (\lambda x.\ \top))$$

$$\vdash \bot = (\forall F.\ F)$$

$$\vdash \neg = (\lambda F.\ F \to \bot)$$

$$\vdash (\wedge) = (\lambda F.\ \lambda G.\ \forall H.\ (F \to G \to H) \to H)$$

$$\vdash (\vee) = (\lambda F.\ \lambda G.\ \forall H.\ (F \to H) \to (G \to H) \to H)$$

Definitions of standard logical symbols

$ex : (\alpha \to bool) \to bool$
Notation: $\exists x.\ F$ abbreviates $ex(\lambda x.\ F)$

$$\vdash ex = (\lambda P.\ \forall G.\ (\forall x.\ (P\ x \to G) \to G))$$

Classical logic

$$\vdash F \lor \neg F$$