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Equational Logic
Summer Term 2012
Exercise Sheet 1
April 20

Homework is due on April 27, before the tutorial.

Exercise 1 (H) (*Bounded Relations*)

A relation $\longrightarrow$ over the set A is called *bounded*, if for each element x, the lengths of all paths from x are bounded. Formally:

$$\forall x \in A.\ \exists n.\ \nexists y \in A.\ x \xrightarrow{n} y$$

Prove or refute:

a) Each terminating relation is bounded.

b) A finitely branching relation is terminating if and only if it is bounded. (Hint: Well-founded induction)

c) Now we call a relation *globally bounded*, if there is a bound that is valid for all elements. Formally:

$$\exists n.\ \forall x \in A.\ \nexists y \in A.\ x \xrightarrow{n} y$$

Prove or refute: Any finitely branching and terminating relation is globally bounded.

Exercise 2 (H) (*Partial Ordering*)

Prove or refute:

a) $\xrightarrow{+}$ is a strict partial order if and only if $\longrightarrow$ is acyclic.

b) $\xrightarrow{*}$ is a partial order if and only if $\longrightarrow$ is acyclic.

Notes: A relation $R \subseteq X \times X$ is called *strict partial order* if it is irreflexive ($\forall x \in X.\ \neg(x\ R\ x)$), transitive ($\forall x, y, z \in X.\ x\ R\ y \wedge y\ R\ z \implies x\ R\ z$), and asymmetric ($\forall x, y \in X.\ x\ R\ y \implies \neg(y\ R\ x)$).

A relation $R \subseteq X \times X$ is called *partial order* if it is reflexive ($\forall x \in X.\ x\ R\ x$), transitive and antisymmetric ($\forall x, y \in X.\ x\ R\ y \wedge y\ R\ x \implies x = y$).

Exercise 3 (T) (*Example*)

Let $(\mathbb{N}_+, \longrightarrow)$ be the reduction system on positive natural numbers, where $\longrightarrow = \{(n, m) \mid 3n = 2m \vee 5n = 7m\}$.

a) Does this system terminate? Justify your answer.

b) Determine the set of all irreducible elements.

c) What is the normal form of 630? Show: $10 \longleftrightarrow 14$ and $20 \xleftrightarrow{*} 63$.

Exercise 4 (T) (*Equivalence Relations*)

A relation $R \subseteq X \times X$ is called *Equivalence relation*, if:

- R is reflexive, i.e. $\forall x \in X.\ x\ R\ x$
- R is transitive, i.e. $\forall x, y, z \in X.\ x\ R\ y \wedge y\ R\ z \implies x\ R\ z$
- R is symmetric, i.e. $\forall x, y \in X.\ x\ R\ y \implies y\ R\ x$

Let $\longrightarrow$ be a relation. Show: $\overset{*}{\longleftrightarrow}$ is the smallest equivalence relation that contains $\longrightarrow$.

Exercise 5 (T) (*Confluence and Normal Form*)

- Show that a reduction system $(A, \longrightarrow)$ is confluent and normalizing, if and only if every element has a unique normal form.
- Give an example of a confluent and normalizing reduction system that does not terminate.