Logics Exercise		
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SS 2017	Exercise Sheet 8	20.06.2017

Submission of homework: Before tutorial on 27.06.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 8.1. **[Decidability]**

1. Resolution for first-order logic is sound and complete.
2. Satisfiability and validity for first-order logic are undecidable.

How do you reconcile these two facts?

Solution:
Resolution gives us a semi-decision procedure for unsatisfiability. That is, if a given formula is not unsatisfiable, it might not terminate. For it to be a decision procedure, it would need to always terminate.

Exercise 8.2. **[Ground Resolution]**
Use ground resolution to prove that the following formula is valid:

$$(\forall x P(x, f(x))) \longrightarrow \exists y P(c, y)$$

Solution:

$$\begin{aligned}
&\neg((\forall x P(x, f(x))) \longrightarrow \exists y P(c, y)) \\
&(\forall x P(x, f(x))) \wedge \neg\exists y P(c, y)) \\
&(\forall x P(x, f(x))) \wedge \forall y \neg P(c, y)) \\
&\forall x \forall y (P(x, f(x)) \wedge \neg P(c, y)) && \text{(Skolem-Form)}
\end{aligned}$$

Now enumerate the Herbrand expansion:

$$E(F) = \{P(c, f(c)) \wedge \neg P(c, f(c)), \ldots\}$$

With resolution, we immediately get $\square$ from the first item in the enumeration.

Exercise 8.3. **[Barber Paradox]**
Consider the following facts:

1. Every barber shaves those who do not shave themselves.
2. No barber shaves anyone who shaves themself.

Show with resolution that there are no barbers by resolution.

Solution:
We model this using the conjunction of the following formula:

- $\forall x(B(x) \longrightarrow (\forall y(\neg S(y,y) \longrightarrow S(x,y))))$
- $\forall x(B(x) \longrightarrow (\forall y(S(y,y) \longrightarrow \neg S(x,y))))$
- $\exists x B(x)$

The predicate symbol B means "is barber" and the predicate symbol S means "shaves". The first two formulas follow directly from the facts. The last formula is used to obtain an unsatisfiable formula, which is what we need for a resolution proof.

The Skolem normal form of this conjunction is:

$$\forall x \forall y((B(x) \longrightarrow \neg S(y,y) \longrightarrow S(x,y)) \wedge (B(x) \longrightarrow S(y,y) \longrightarrow \neg S(x,y)) \wedge B(c))$$

This can be translated into clauses easily (we have already renamed variables here):

$$\{B(c)\}, \{\neg B(x), \neg S(y,y), \neg S(x,y)\}, \{\neg B(z), S(w,w), S(z,w)\}$$

Resolution:

1. $\{B(c)\}$
2. $\{\neg B(x), \neg S(y,y), \neg S(x,y)\}$
3. $\{\neg B(z), S(w,w), S(z,w)\}$
4. $\{\neg S(y,y), \neg S(c,y)\}$ (1, 2, with $\{x \mapsto c\}$)
5. $\{S(w,w), S(c,w)\}$ (1, 3, with $\{z \mapsto c\}$)
6. $\square$ (4, 5, with $\{y \mapsto c, w \mapsto c\}$)

Homework 8.1. [Restricted Resolution] (8 points)
In the resolution procedure as defined in the lecture slides, we can unify arbitrarily many literals from two clauses. Consider a modified resolution procedure, where exactly one literal is picked. We add another rule ("collapsing rule"): For a clause $C = \{L_1, \ldots, L_n\}$, where $\{L_i, L_j\}$ can be unified using a mgu δ, add another clause $C' = (C - L_i)\delta$.

For example, given the clause

$$C = \{\neg W(x), \neg W(f(y)), T(x, y), \neg W(f(c))\}$$

we can apply the collapsing rule as follows:

$$L_1 = \neg W(x), L_2 = \neg W(f(y)), \delta = \{x \mapsto f(y)\}, C' = \{\neg W(f(y)), T(f(y), y), \neg W(f(c))\}$$

(Note that there are multiple possible ways to apply the collapsing rule to C.)

Prove that our modified resolution calculus, including collapsing rule, can be simulated by the original resolution calculus, and vice versa.

Homework 8.2. [Resolution] (8 points)
Show with resolution that:

1. $\forall x(\neg R(x) \longrightarrow R(f(x))) \longrightarrow \exists x(R(x) \wedge R(f(f(x))))$ is valid
2. $\exists x(P(x) \wedge \neg P(f(f(x)))) \wedge \forall x(P(x) \longrightarrow P(f(x)))$ is unsatisfiable

Homework 8.3. [Equality] (4 points)
We consider how to model equality in predicate logic. In the lecture slides, the following axiom schema for congruence is used:

$$\frac{Eq(x_i, y)}{Eq(f(x_1, \ldots, x_i, \ldots, x_n), f(x_1, \ldots, y, \ldots, x_n))}$$

Assume that this schema is replaced by:

$$\frac{Eq(x_1, y_1) \quad \cdots \quad Eq(x_n, y_n)}{Eq(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n))}$$

Reflexivity, symmetry and transitivity stay unchanged. Show that the above modified schemas is equivalent to the schemas from the lecture.

Hint: Simulate the modified schema with the original one and vice versa.