First-Order Logic
Herbrand Theory

Herbrand universe

The Herbrand universe $T(F)$ of a closed formula F in Skolem form is the set of all terms that can be constructed using the function symbols in F.

In the special case that F contains no constants, we first pick an arbitrary constant, say a, and then construct the terms.

Formally, $T(F)$ is inductively defined as follows:

- All constants occurring in F belong to $T(F)$;
  if no constant occurs in F, then $a \in T(F)$
  where a is some arbitrary constant.
- For every n-ary function symbol f occurring in F,
  if $t_1, t_2, \ldots, t_n \in T(F)$ then $f(t_1, t_2, \ldots, t_n) \in T(F)$.

Note: All terms in $T(F)$ are variable-free by construction!

Example

$F = \forall x \forall y \; P(f(x), g(c, y))$

Herbrand structure

Let F be a closed formula in Skolem form.
A structure $\mathcal{A}$ suitable for F is a Herbrand structure for F if it satisfies the following conditions:

- $U_{\mathcal{A}} = T(F)$, and
- for every n-ary function symbol f occurring in F and every $t_1, \ldots, t_n \in T(F)$: $f^{\mathcal{A}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

Fact

If $\mathcal{A}$ is a Herbrand structure, then $\mathcal{A}(t) = t$ for all $t \in U_{\mathcal{A}}$.

We call a Herbrand structure that is a model a Herbrand model.

Matrix of a formula

Definition

The matrix of a formula F is the result of removing all quantifiers (all $\forall x$ and $\exists x$) from F. The matrix is denoted by F^*.

Fundamental theorem of predicate logic

Theorem

Let F be a closed formula in Skolem form.
Then F is satisfiable iff it has a Herbrand model.

Proof If F has a Herbrand model then it is satisfiable.

For the other direction let $\mathcal{A}$ be an arbitrary model of F.
We define a Herbrand structure $\mathcal{T}$ as follows:

Universe	$U_{\mathcal{T}} = T(F)$
Function symbols	$f^{\mathcal{T}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$

If F contains no constant: $a^{\mathcal{A}} = u$ for some arbitrary $u \in U_{\mathcal{A}}$

Predicate symbols	$(t_1, \ldots, t_n) \in P^{\mathcal{T}}$ iff $(\mathcal{A}(t_1), \ldots, \mathcal{A}(t_n)) \in P^{\mathcal{A}}$

Claim: $\mathcal{T}$ is also a model of F.

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We prove a stronger assertion:

> *For every closed formula G in Skolem form*
> *such that all fun. and pred. symbols in G occur in F:*
> *if $\mathcal{A} \models G$ then $\mathcal{T} \models G$*

Proof By induction on the number n of universal quantifiers of G.

Basis $n = 0$. Then G has no quantifiers at all.
Therefore $\mathcal{A}(G) = \mathcal{T}(G)$ (why?), and we are done.

Induction step: $G = \forall x\ H$.

$$
\begin{array}{lll}
 & \mathcal{A} \models G & \\
\Rightarrow & \text{for every } u \in U_{\mathcal{A}}\text{: } \mathcal{A}[u/x](H) = 1 & \\
\Rightarrow & \text{for every } u \in U_{\mathcal{A}} \text{ of the form } u = \mathcal{A}(t) & \\
 & \quad \text{where } t \in T(G)\text{: } \mathcal{A}[u/x](H) = 1 & \\
\Rightarrow & \text{for every } t \in T(G)\text{: } \mathcal{A}[\mathcal{A}(t)/x](H) = 1 & \\
\Rightarrow & \text{for every } t \in T(G)\text{: } \mathcal{A}(H[t/x]) = 1 & \text{(substitution lemma)} \\
\Rightarrow & \text{for every } t \in T(G)\text{: } \mathcal{T}(H[t/x]) = 1 & \text{(induction hypothesis)} \\
\Rightarrow & \text{for every } t \in T(G)\text{: } \mathcal{T}[\mathcal{T}(t)/x](H) = 1 & \text{(substitution lemma)} \\
\Rightarrow & \text{for every } t \in T(G)\text{: } \mathcal{T}[t/x](H) = 1 & (\mathcal{T} \text{ is Herbrand structure)} \\
\Rightarrow & \mathcal{T}(\forall x\ H) = 1 & (U_{\mathcal{T}} = T(G)) \\
\Rightarrow & \mathcal{T} \models G &
\end{array}
$$

Theorem

Let F be a closed formula in Skolem form.
Then F is satisfiable iff it has a Herbrand model.

What goes wrong if F is not closed or not in Skolem form?

Herbrand expansion

Let $F = \forall y_1 \ldots \forall y_n F^*$ be a closed formula in Skolem form.
The Herbrand expansion of F is the set of formulas

$$E(F) = \{F^*[t_1/y_1] \ldots [t_n/y_n] \mid t_1, \ldots, t_n \in T(F)\}$$

Informally: the formulas of $E(F)$ are the result of substituting terms from $T(F)$ for the variables of F^* in every possible way.

Example

$E(\forall x \forall y\ P(f(x), g(c, y)) =$

Note The Herbrand expansion can be viewed as a set of propositional formulas.

Gödel-Herbrand-Skolem Theorem

Theorem

Let F be a closed formula in Skolem form.
Then F is satisfiable iff its Herbrand expansion $E(F)$ is satisfiable (in the sense of propositional logic).

Proof By the fundamental theorem, it suffices to show:
F has a Herbrand model iff $E(F)$ is satisfiable.

Let $F = \forall y_1 \ldots \forall y_n F^*$.

$\mathcal{A}$ is a Herbrand model of F
iff for all $t_1, \ldots, t_n \in T(F)$, $\mathcal{A}[t_1/y_1] \ldots [t_n/y_n](F^*) = 1$
iff for all $t_1, \ldots, t_n \in T(F)$, $\mathcal{A}(F^*[t_1/y_1] \ldots [t_n/y_n]) = 1$
iff for all $G \in E(F)$, $\mathcal{A}(G) = 1$
iff $\mathcal{A}$ is a model of $E(F)$

Herbrand's Theorem

Theorem

Let F be a closed formula in Skolem form.
F is unsatisfiable iff some finite subset of $E(F)$ is unsatisfiable.

Proof Follows immediately from the Gödel-Herbrand-Skolem Theorem and the Compactness Theorem.

Gilmore's Algorithm

Let F be a closed formula in Skolem form
and let $F_1, F_2, F_3, \ldots$ be an computable enumeration of $E(F)$.

Input: F

$n := 0$;
repeat $n := n + 1$;
until $(F_1 \wedge F_2 \wedge \ldots \wedge F_n)$ is unsatisfiable;
return "unsatisfiable"

The algorithm terminates iff F is unsatisfiable.

Semi-decidiability Theorems

Theorem

(a) *The unsatisfiability problem of predicate logic is (only) semi-decidable.*

(b) *The validity problem of predicate logic is (only) semi-decidable.*

Proof

(a) Gilmore's algorithm is a semi-decision procedure.
(The problem is undecidable. Proof later)
(b) F valid iff $\neg F$ unsatisfiable.

Löwenheim-Skolem Theorem

Theorem

Every satisfiable formula of first-order predicate logic has a model with a countable universe.

Proof Let F be a formula,
and let G be an equisatisfiable formula in Skolem form
(as produced by the Normal Form transformations).
Fact: Every model of G is a model of F. (Check this!)

$$\begin{aligned}
F \text{ satisfiable} \quad &\Rightarrow \quad G \text{ satisfiable} \\
&\Rightarrow \quad G \text{ has a Herbrand model } \mathcal{T} \\
&\Rightarrow \quad F \text{ also has that model } \mathcal{T} \\
&\Rightarrow \quad F \text{ has a countable model} \\
& \quad \text{(Herbrand universes are countable)}
\end{aligned}$$

Löwenheim-Skolem Theorem

Formulas of first-order logic cannot enforce uncountable models

Formulas of first-order logic cannot axiomatize the real numbers
because there will always be countable models