First-Order Logic
Basic Proof Theory

Substitution

Substitution $F[t/x]$ assumes that bound variables in F are automatically renamed to avoid capturing free variables in t.

Example

$(\forall x \, P(x, y))[f(x)/y] \; = \forall x' \, P(x', f(x))$

Sequent Calculus rules

$$\frac{F[t/x], \forall x\, F, \Gamma \Rightarrow \Delta}{\forall x\, F, \Gamma \Rightarrow \Delta}\ \forall L \qquad \frac{\Gamma \Rightarrow F[y/x], \Delta}{\Gamma \Rightarrow \forall x\, F, \Delta}\ \forall R(*)$$

$$\frac{F[y/x], \Gamma \Rightarrow \Delta}{\exists x\, F, \Gamma \Rightarrow \Delta}\ \exists L(*) \qquad \frac{\Gamma \Rightarrow F[t/x], \exists x\, F, \Delta}{\Gamma \Rightarrow \exists x\, F, \Delta}\ \exists R$$

$(*)$: y not free in the conclusion of the rule

Note: $\forall L$ and $\exists R$ do not delete the principal formula

Natural Deduction rules

$$\frac{F[y/x]}{\forall x\, F}\ \forall I(*) \qquad \frac{\forall x\, F}{F[t/x]}\ \forall E$$

$$\frac{F[t/x]}{\exists x\, F}\ \exists I \qquad \frac{\exists x\, F \qquad \begin{matrix}[F[y/x]] \\ \vdots \\ H\end{matrix}}{H}\ \exists E(**)$$

$(*)$: $(y = x$ or $y \notin fv(F))$ and
y not free in an open assumption in the proof of $F[y/x]$

$(**)$: $(y = x$ or $y \notin fv(F))$ and
y not free in H or in an open assumption in the proof of the second premise, except for $F[y/x]$

Hilbert System

Additional rule $\forall I$:
if F is provable then $\forall y\, F[y/x]$ is provable
provided y not free in the assumptions and ($y = x$ or $y \notin fv(F)$)

Additional axioms:

$\forall x\, F \rightarrow F[t/x]$

$F[t/x] \rightarrow \exists x\, F$

$\forall x(G \rightarrow F) \rightarrow (G \rightarrow \forall y\, F[y/x])$ if $y \notin fv(G), y = x$ or $y \notin fv(F)$

$\forall x(F \rightarrow G) \rightarrow (\exists y\, F[y/x] \rightarrow G)$ if $y \notin fv(G), y = x$ or $y \notin fv(F)$