First-Order Logic
Resolution

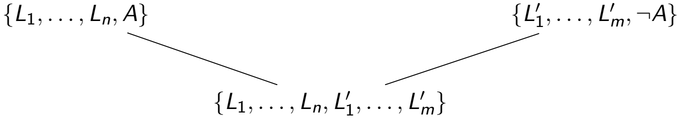
Resolution for predicate logic

Gilmore's algorithm is correct and complete,
but useless in practice.

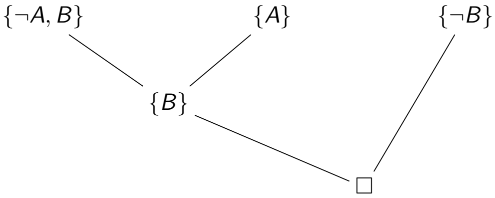
We upgrade resolution to make it work for predicate logic.

Recall: resolution in propositional logic

Resolution step:



Resolution graph:



A set of clauses is unsatisfiable iff the empty clause can be derived.

Adapting Gilmore's Algorithm

Gilmore's Algorithm:

Let F be a closed formula in Skolem form
and let $F_1, F_2, F_3, \ldots$ be an enumeration of $E(F)$.

$n := 0$;
repeat $n := n + 1$
until $(F_1 \wedge F_2 \wedge \ldots \wedge F_n)$ is unsatisfiable;
   – *this can be checked with any calculus for propositional logic*
return "unsatisfiable"

"any calculus" $\rightsquigarrow$ use resolution for the unsatisfiability test

Terminology

Literal/clause/CNF is defined as for propositional logic
but with the atomic formulas of predicate logic.

A ground term/formula/etc is a term/formula/etc
that does not contain any variables.

An instance of a term/formula/etc
is the result of applying a substitution to a term/formula/etc.

A ground instance
is an instance that does not contain any variables.

Clause Herbrand expansion

Let $F = \forall y_1 \ldots \forall y_n\, F^*$ be a closed formula in Skolem form with F^* in CNF, and let $C_1, \ldots, C_m$ be the clauses of F^*.

The clause Herbrand expansion of F is the set of ground clauses

$$CE(F) = \bigcup_{i=1}^{m} \{\, C_i[t_1/y_1] \ldots [t_n/y_n] \mid t_1, \ldots, t_n \in T(F)\}$$

Lemma

$CE(F)$ is unsatisfiable iff $E(F)$ is unsatisfiable.

Proof Informally speaking, "$CE(F) \equiv E(F)$".

Ground resolution algorithm

Let F be a closed formula in Skolem form with F^* in CNF.

Let $C_1, C_2, C_3, \ldots$ be an enumeration of $CE(F)$.

$n := 0$;
$S := \emptyset$;
repeat
  $n := n + 1$;
  $S := S \cup \{C_n\}$;
until $S \vdash_{Res} \Box$

return "unsatisfiable"

Note: The search for $\Box$ can be performed incrementally every time S is extended.

Example

$F^* = \{\{\neg P(x), \neg P(f(a)), Q(y)\},\ \{P(y)\},\ \{\neg P(g(b,x)), \neg Q(b)\}\}$

Ground resolution theorem

The correctness of the ground resolution algorithm can be rephrased as follows:

Theorem

A formula $F = \forall y_1 \ldots \forall y_n\ F^*$ with F^* in CNF is unsatisfiable iff there is a sequence of ground clauses $C_1, \ldots, C_m = \Box$ such that for every $i = 1, \ldots, m$

- *either C_i is a ground instance of a clause $C \in F^*$, i.e. $C_i = C[t_1/y_1] \ldots [t_n/y_n]$ where $t_1, \ldots, t_n \in T(F)$,*
- *or C_i is a resolvent of two clauses C_a, C_b with $a < i$ and $b < i$*

Where do the ground substitutions come from?

Better:

- allow substitutions with variables
- only instantiate clauses enough to allow one (new kind of) resolution step

Example

Resolve $\{P(x), Q(x)\}$ and $\{\neg P(f(y)), R(y)\}$

Substitutions as functions

Substitutions are functions from variables to terms:
$[t/x]$ maps x to t (and all other variales to themselves)

Functions can be composed.

Composition of substitutions is denoted by juxtaposition:
$[t_1/x][t_2/y]$ first substitutes t_1 for x and then substitutes t_2 for y.

Example

$$(P(x, y))[f(y)/x][b/y] \;=\; (P(f(y), y))[b/y] \;=\; P(f(b), b)$$

Similarly we can compose arbitrary substitutions σ_1 and σ_2:
$\sigma_1\sigma_2$ is the substitution that applies σ_1 first and then σ_2.

Substitutions are functions. Therefore

$$\sigma_1 = \sigma_2 \quad \text{iff} \quad \text{for all variables } x,\ x\sigma_1 = x\sigma_2$$

Substitutions as functions

Definition

The domain of a substitution: $dom(\sigma) = \{x \mid x\sigma \neq x\}$

Example

$dom([a/x][b/y]) = \{x, y\}$

Substitutions are defined to have finite domain.
Therefore every substitution can be written as a simultaneous substitution $[t_1/x_1, \ldots, t_n/x_n]$.

Unifier and most general unifier

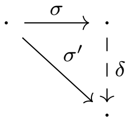
Let $\mathbf{L} = \{L_1, \ldots, L_k\}$ be a set of literals.

A substitution σ is a unifier of $\mathbf{L}$ if

$$L_1\sigma = L_2\sigma = \cdots = L_k\sigma$$

i.e. if $|\mathbf{L}\sigma| = 1$, where $\mathbf{L}\sigma = \{L_1\sigma, \ldots, L_k\sigma\}$.

A unifier σ of $\mathbf{L}$ is a most general unifier (mgu) of $\mathbf{L}$ if
for every unifier σ' of $\mathbf{L}$ there is a substitution δ such that $\sigma' = \sigma\delta$.



Exercise

Unifiable?			Yes	No
	$P(f(x))$	$P(g(y))$		x
	$P(x)$	$P(f(y))$	x	
	$P(x)$	$P(f(x))$		x
	$P(x, f(y))$	$P(f(u), f(z))$	x	
	$P(x, f(x))$	$P(f(y), y)$		x
	$P(x, g(x), g^2(x))$	$P(f(z), w, g(w))$	x	
$P(x, f(y))$	$P(g(y), f(a))$	$P(g(a), z)$	x	

Unification algorithm

Input: a set $\mathbf{L} \neq \emptyset$ of literals

$\sigma := []$ (the empty substitution)

while $|\mathbf{L}\sigma| > 1$ **do**

  Find the first position at which two literals $L_1, L_2 \in \mathbf{L}\sigma$ differ

  if none of the two characters at that position is a variable

  then return "non-unifiable"

  else let x be the variable and t the term starting at that position

    if x occurs in t

    then return "non-unifiable"

    else $\sigma := \sigma\,[t/x]$

return σ

Example

$\{\ \neg P(f(z, g(a, y)), h(z)),$
$\ \neg P(f(f(u, v), w), h(f(a, b)))\ \}$

Correctness of the unification algorithm

Lemma

The unification algorithm terminates.

Proof Every iteration of the **while**-loop (possibly except the last) replaces a variable x by a term t not containing x, and so the number of variables occurring in $\mathbf{L}\sigma$ decreases by one.

Lemma

If $\mathbf{L}$ *is non-unifiable then the algorithm returns "non-unifiable".*

Proof If $\mathbf{L}$ is non-unifiable then the algorithm can never exit the loop normally.

Correctness/completeness of the unification algorithm

Lemma

*If **L** is unifiable then the algorithm returns the mgu of **L** (and so in particular every unifiable set **L** has an mgu).*

Proof Assume **L** is unifiable and let n be the number of iterations of the loop on input **L**.

Let $\sigma_0 = []$, for $1 \leq i \leq n$ let σ_i be the value of σ after the i-th iteration of the loop.

We prove for every $0 \leq i \leq n$:

(a) If $1 \leq i$, the i-th iteration does not return "non-unifiable".

(b) For every unifier σ' of **L** there is a substitution δ_i such that $\sigma' = \sigma_i \, \delta_i$.

By (a) the algorithm exits the loop normally after n iterations.
By (b) it returns a most general unifier.

Correctness/completeness of the unification algorithm

Proof of (a) and (b) by induction on i:

Basis ($i = 0$): For (a) there is nothing to prove.
For (b) take $\delta_0 = \sigma'$.

Step ($i \Rightarrow i + 1$)

For (a), since $|\mathbf{L}\sigma_i| > 1$ and $\mathbf{L}\sigma_i$ unifiable, x and t exist and x does not occur in t, and so "non-unifiable" is not returned.

For (b): Let σ' be a unifier of $\mathbf{L}$. IH: $\sigma' = \sigma_i\delta_i$ for some δ_i.
δ_i must be of the form $[t_1/x_1, \ldots, t_k/x_k, u/x]$ where $x_1, \ldots, x_k, x$ are distinct. Define $\delta_{i+1} = [t_1/x_1, \ldots, t_k/x_k]$.
Note $u = x\delta_i = t\delta_i = t\delta_{i+1}$ ($\sigma_i\delta_i$ is unifier (IH), x not in t)

$$
\begin{array}{lll}
 & \sigma_{i+1}\,\delta_{i+1} & \\
= & \sigma_i\,[t/x]\,\delta_{i+1} & \text{(algorithm extends } \sigma_i \text{ with } [t/x]) \\
= & \sigma_i\,[t_1/x_1, \ldots, t_k/x_k, t\delta_{i+1}/x] & \\
= & \sigma_i\,[t_1/x_1, \ldots, t_k/x_k, u/x] & \text{(Note } u = t\delta_{i+1}) \\
= & \sigma_i\,\delta_i & \\
= & \sigma' & \text{(IH)}
\end{array}
$$

The standard view of unification

A unification problem is a pair of terms $s =^? t$
(or a set of pairs $\{s_1 =^? t_1, \ldots, s_n =^? t_n\}$)

A unifier is a substitution σ such that $s\sigma = t\sigma$
(or $s_1\sigma = t_1\sigma, \ldots, s_n\sigma = t_n\sigma$)

Renaming

Definition

A substitution ρ is a renaming if for every variable x, $x\rho$ is a variable and ρ is injective on $dom(\rho)$.

Resolvents for first-order logic

A clause R is a resolvent of two clauses C_1 and C_2 if the following holds:

- There is a renaming ρ such that
  no variable occurs in both C_1 and $C_2\,\rho$ and
  ρ is injective on the set of variables in C_2
- There are literals $L_1, \ldots, L_m \in C_1$ $(m \geq 1)$
  and literals $L'_1, \ldots, L'_n \in C_2\,\rho$ $(n \geq 1)$ such that

  $$\mathbf{L} = \{\overline{L_1}, \ldots, \overline{L_m}, L'_1, \ldots, L'_n\}$$

  is unifiable. Let σ be an mgu of $\mathbf{L}$.
- $R = ((C_1 - \{L_1, \ldots, L_m\}) \cup (C_2\,\rho - \{L'_1, \ldots, L'_n\}))\sigma$

Example

$C_1 = \{\ P(x),\ Q(x),\ P(g(y))\ \}$ and $C_2 = \{\ \neg P(x),\ R(f(x), a)\ \}$

Exercise

How many resolvents are there?

C_1	C_2	Resolvents
$\{P(x), Q(x, y)\}$	$\{\neg P(f(x))\}$	
$\{Q(g(x)), R(f(x))\}$	$\{\neg Q(f(x))\}$	
$\{P(x), P(f(x))\}$	$\{\neg P(y), Q(y, z)\}$	

Why renaming?

Example

$\forall x(P(x) \land \neg P(f(x)))$

Resolution for first-order logic

As for propositional logic, $F \vdash_{Res} C$ means that clause C can be derived from a set of clauses F by a sequence of resolution steps, i.e. that there is a sequence of clauses $C_1, \ldots, C_m = C$ such that for every C_i

- either $C_i \in F$
- or C_i is the resolvent of C_a and C_b where $a, b < i$.

Questions:

Correctness Does $F \vdash_{Res} \square$ imply that F is unsatisfiable?

Completeness Does unsatisfiability of F imply $F \vdash_{Res} \square$?

Exercise

Derive $\Box$ from the following clauses:

1. $\{\neg P(x), Q(x), R(x, f(x))\}$
2. $\{\neg P(x), Q(x), S(f(x))\}$
3. $\{T(a)\}$
4. $\{P(a)\}$
5. $\{\neg R(a, z), T(z)\}$
6. $\{\neg T(x), \neg Q(x)\}$
7. $\{\neg T(y), \neg S(y)\}$

Correctness of Resolution for First-Order Logic

Definition

The universal closure of a formula H with free variables $x_1, \ldots, x_n$:

$$\forall H \; = \; \forall x_1 \forall x_2 \ldots \forall x_n H$$

Theorem

Let F be a closed formula in Skolem form with matrix F^* in CNF. If $F^* \vdash_{Res} \Box$ then F is unsatisfiable.

Completeness: The idea

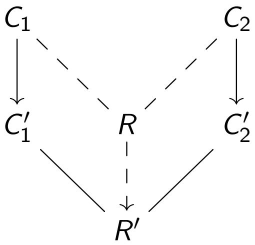
Simulate ground resolution because that is complete

Lift the resolution proof from the ground resolution proof

Lifting Lemma

Let C_1, C_2 be two clauses and let C_1', C_2' be two ground instances with (propositional) resolvent R'.

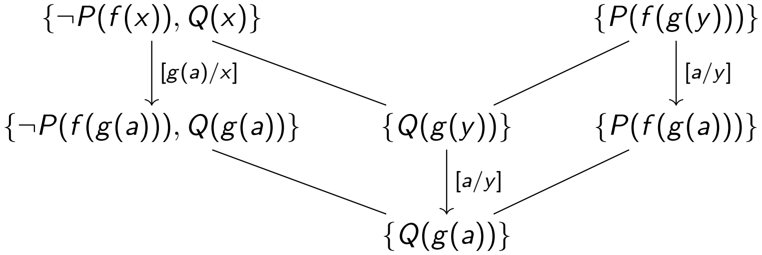
Then there is a resolvent R of C_1, C_2 such that R' is a ground instance of R.



$\rightarrow$: Substitution

—: Resolution

Lifting Lemma: example



Completeness of Resolution for First-Order Logic

Theorem

Let F be a closed formula in Skolem form with matrix F^* in CNF.
If F is unsatisfiable then $F^* \vdash_{Res} \Box$.

Proof If F is unsatisfiable, there is a ground resolution proof $C'_1, \dots, C'_n = \Box$. We transform this step by step into a resolution proof $C_1, \dots, C_n = \Box$ such that C'_i is a ground instance of C_i.

If C'_i is a ground instance of some clause $C \in F^*$:
Set $C_i = C$

If C'_i is a resolvent of C'_a, C'_b $(a, b < i)$:
C'_a, C'_b have been transformed already into C_a, C_b s.t. C'_a, C'_b are ground instances of C_a, C_b. By the Lifting Lemma there is a resolvent R of C_a, C_b s.t. C'_i is a ground instance of R.
Set $C_i = R$.

Resolution Theorem for First-Order Logic

Theorem

Let F be a closed formula in Skolem form with matrix F^* in CNF.
Then F is unsatisfiable iff $F^* \vdash_{Res} \square$.

A resolution algorithm

Input: A closed formula F in Skolem form with matrix S in CNF, i.e. S is a finite set of clauses

while $\Box \notin S$ and
there are clauses $C_a, C_b \in S$ and resolvent R of C_a and C_b
such that $R \notin S$ (modulo renaming)
do $S := S \cup \{R\}$

The selection of resolvents must be *fair:*
every resolvent is added eventually

Three possible behaviours:

- The algorithm terminates and $\Box \in S$
  $\Rightarrow F$ is unsatisfiable
- The algorithm terminates and $\Box \notin S$
  $\Rightarrow F$ is satisfiable
- The algorithm does not terminate
  ($\Rightarrow F$ is satisfiable)

Refinements of resolution

Problems of resolution:

- Branching degree of the search space too large
- Too many dead ends
- Combinatorial explosion of the search space

Solution:
Strategies and heuristics: forbid certain resolution steps, which narrows the search space.

But: Completeness must be preserved!