First-order Predicate Logic
Theories

Definitions

Definition

A signature Σ is a set of predicate and function symbols.

A Σ-formula is a formula that contains only predicate and function symbols from Σ.

A Σ-structure is a structure that interprets all predicate and function symbols from Σ.

Definition

A sentence is a closed formula.

In the sequel, S is a set of sentences.

Theories

Definition

A theory is a set of sentences S such that S is closed under consequence: If $S \models F$ and F is closed, then $F \in S$.

Let $\mathcal{A}$ be a Σ-structure:
$Th(\mathcal{A})$ is the set of all sentences true in $\mathcal{A}$:
$Th(\mathcal{A}) = \{F \mid F \ \Sigma\text{-sentence and } \mathcal{A} \models F\}$

Lemma

Let $\mathcal{A}$ be a Σ-structure and F a Σ-sentence.
Then $\mathcal{A} \models F$ iff $Th(\mathcal{A}) \models F$.

Corollary

$Th(\mathcal{A})$ is a theory.

Example

Notation: $(\mathbb{Z}, +, \leq)$ denotes the structure with universe $\mathbb{Z}$ and the standard interpretations for the symbols $+$ and $\leq$.
The same notation is used for other standard structures where the interpretation of a symbol is clear from the symbol.

Example (Linear integer arithmetic)

$Th(\mathbb{Z}, +, \leq)$ is the set of all sentences over the signature $\{+, \leq\}$ that are true in the structure $(\mathbb{Z}, +, \leq)$.

Axioms and consequences

Definition

Let S be a set of Σ-sentences.

$Cn(S)$ is the set of consequences of S:
$Cn(S) = \{F \mid F\ \Sigma\text{-sentence and } S \models F\}$

A theory T is axiomatized by S if $T = Cn(S)$

A theory T is axiomatizable if there is some decidable or recursively enumerable S that axiomatizes T.

A theory T is finitely axiomatizable
if there is some finite S that axiomatizes T.

Examples

$Cn(\emptyset)$ is the set of valid sentences.
$Cn(\{\forall x \forall y \forall z\ (x * y) * z = x * (y * z)\})$ is the set of sentences that are true in all semigroups.

Famous numerical theories

$Th(\mathbb{R}, +, \leq)$ is called linear real arithmetic.
It is decidable.

$Th(\mathbb{R}, +, *, \leq)$ is called real arithmetic.
It is decidable.

$Th(\mathbb{Z}, +, \leq)$ is called linear integer arithmetic or Presburger arithmetic.
It is decidable.

$Th(\mathbb{Z}, +, *, \leq)$ is called integer arithmetic.
It is not even semidecidable ($=$ r.e.).

Decidability via special algorithms.

Completeness and elementary equivalence

Definition

A theory T is complete if for every sentence F, $T \models F$ or $T \models \neg F$.

Definition

Two structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if $Th(\mathcal{A}) = Th(\mathcal{B})$.

Theorem

A theory T is complete iff all its models are elementarily equivalent.