Logics Exercise		
	TU München Institut für Informatik	Prof. Tobias Nipkow Lars Hupel
SS 2018	Exercise Sheet 5	08.05.2018

Submission of homework: Before tutorial on 15.05.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 5.1. [Induction Proofs]
Show that the following schema has a proof in natural deduction for all $n \geq 1$:

$$P_n = ((A_1 \wedge (A_2 \wedge (\ldots \wedge A_n) \ldots) \to B) \to (A_1 \to (A_2 \to (\ldots (A_n \to B) \ldots))))$$

Solution:

- Base case: $n = 1$

  Formula: $P_1 = (A_1 \to B) \to (A_1 \to B)$

  Proof: trivial through $\to I$

- Induction step: $n \rightsquigarrow n + 1$

$$\dfrac{\dfrac{\dfrac{\text{(IH)}^{1}\ \dfrac{}{(A_2 \wedge \ldots \wedge A_{n+1} \to B) \to (A_2 \to \ldots \to A_{n+1} \to B)} \qquad \dfrac{\overline{B}\ \text{(subproof)}}{A_2 \wedge \ldots \wedge A_{n+1} \to B}\ {\to} I_3}{A_2 \to \ldots \to A_{n+1} \to B}\ {\to} E}{A_1 \to A_2 \to \ldots \to A_{n+1} \to B}\ {\to} I_2}{(A_1 \wedge A_2 \wedge \ldots \wedge A_{n+1} \to B) \to (A_1 \to A_2 \to \ldots \to A_{n+1} \to B)}\ {\to} I_1$$

Subproof:

$$\dfrac{\begin{array}{c}[A_1 \wedge A_2 \wedge \ldots \wedge A_{n+1} \to B]^1 \\ \vdots \\ A_1 \wedge A_2 \wedge \ldots \wedge A_{n+1} \to B\end{array} \qquad \dfrac{\begin{array}{c}[A_1]^2 \\ \vdots \\ A_1\end{array} \qquad \begin{array}{c}[A_2 \wedge \ldots \wedge A_{n+1}]^3 \\ \vdots \\ A_2 \wedge \ldots \wedge A_{n+1}\end{array}}{A_1 \wedge A_2 \wedge \ldots \wedge A_{n+1}}\ \wedge I}{B}\ {\to} E$$

[1] *Note:* We use the induction hypothesis with the atomic formulas $\{A_2, \ldots, A_{n+1}\}$.

Exercise 5.2. [From Sequent Calculus to Natural Deduction]
How can we construct, from a sequent calculus proof $\Gamma \Rightarrow \Delta$, a natural deduction proof $\Gamma \vdash_{\mathrm{N}} \bigvee \Delta$?

Solution:

1. Known from lecture: $\Gamma \Rightarrow \Delta \rightsquigarrow \Gamma, \neg \bigvee \Delta \vdash_{\mathrm{N}} \bot$
2. Use the $\bot$ rule: $\Gamma, \neg \bigvee \Delta \vdash_{\mathrm{N}} \bot \rightsquigarrow \Gamma \vdash_{\mathrm{N}} \bigvee \Delta$

Exercise 5.3. [Hilbert Calculus]
Prove the following formula with a linear proof in Hilbert calculus: $(F \wedge G) \to (G \wedge F)$

Hint: Use the deduction theorem.

Solution:
We first apply the deduction theorem. It remains to construct a proof of $F \wedge G \vdash_H G \wedge H$.

1. $F \wedge G \to G$	A4
2. $F \wedge G$	Γ
3. $F \wedge G \to F$	A3
4. $G \to F \to G \wedge F$	A5
5. G	1, 2
6. $F \to G \wedge F$	4, 5
7. F	3, 2
8. $G \wedge F$	6, 7

Homework 5.1. **[A Smaller Set of Axioms for Hilbert Calculus]** (5 points)
In the lecture, Hilbert calculus for propositional logic was introduced by means of nine axioms. However, the following three axioms are already sufficient:

A1 $F \to (G \to F)$

A2 $(F \to G \to H) \to (F \to G) \to F \to H$

A10 $(\neg F \to \neg G) \to (G \to F)$

Derive the following statement from the axioms above with help of $\to_E$: $\neg(F \to F) \to G$

Homework 5.2. **[From Sequent Calculus to Natural Deduction]** (8 points)
In Exercise 5.2, we constructed, from a sequent calculus proof of $\Gamma \Rightarrow \Delta$, a natural deduction proof $\Gamma \vdash_{\mathrm{N}} \bigvee \Delta$. That construction was "classical", because it required the use of the $\perp$ rule. We will consider a restricted Sequent Calculus here and the special case where $\Delta = \{F\}$, i.e., always contains exactly one formula.

The following system – a variant of Sequent Calculus – is called **G3c:**

$$\text{Ax}\quad P, \Gamma \Rightarrow P \; (P \text{ atomic}) \qquad\qquad \text{L}\perp\quad \perp, \Gamma \Rightarrow A$$

$$\text{L}\wedge\ \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad\qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \qquad\qquad \text{R}\vee\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ (i = 0, 1)$$

$$\text{L}\to\ \frac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C} \qquad\qquad \text{R}\to\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

Give a direct construction (without using the $\perp$ rule, but $\perp E$ is allowed) that transforms a G3c proof $\Gamma \Rightarrow F$ into a natural deduction proof $\Gamma \vdash_{\mathrm{N}} F$.

Hints: Perform an induction on the depth of the G3c proof. It is sufficient to consider the base cases and two other cases of your own choosing.

Homework 5.3. **[Truth Tables]** (7 points)
In the lecture, the following lemma was discussed:

Let atoms$(F) \subseteq \{A_1, \ldots, A_n\}$. Then we can construct a proof $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_{\mathrm{N}} F^{\mathcal{A}}$.

Recall the definition of $F^{\mathcal{A}}$:

$$F^{\mathcal{A}} = \begin{cases} F & \text{if } \mathcal{A}(F) = 1 \\ \neg F & \text{otherwise} \end{cases}$$

Prove the lemma by induction on F! It suffices to prove the cases for atomic formulas, negation, and disjunction.