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SS 2018 Exercise Sheet 7 23.05.2018

Submission of homework: Before tutorial on 29.05.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 7.1. [Herbrand Models]
Given the formula

$$F = \forall x \forall y (P(f(x), g(y)) \land \neg P(g(x), f(y)))$$

1. Specify a Herbrand model for F.
2. Specify a Herbrand structure suitable for F, which is not a model of F.

Solution:
We define $U_{\mathcal{A}} = T(F)$, i.e., the Herbrand universe for F. We invent a constant $a \in T(F)$. We define $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ to be the Herbrand interpretations.

1. $P^{\mathcal{A}} = \{(f(t_1), g(t_2)) \mid t_1, t_2 \in T(F)\}$.
2. $P^{\mathcal{A}} = \{(g(t_1), f(t_2)) \mid t_1, t_2 \in T(F)\}$.

Exercise 7.2. [(In)finite Models]

1. Show that any model (for a formula of predicate logic) with a universe of size n can be extended to a model of size m for any $m \geq n$. Can it also be extended to an *infinite* model?
2. Now consider the extension of predicate logic with equality. Does above property still hold?

Solution:

1. Let $\mathcal{A}$ be a model. We pick any $d \in U_{\mathcal{A}}$ as an element to "clone" $m - n$ times.

   The precise construction works as follows: We define $D = \{(d, k) \mid k \in \mathbb{N} \wedge k < m - n\}$. Now, we extend $U_{\mathcal{A}}$ with D.

   Let $\mathcal{A}'$ be a structure with the universe $U_{\mathcal{A}'} = U_{\mathcal{A}} \uplus D$. All functions and predicate symbols are interpreted identically to $\mathcal{A}$, with the extension that all elements (d, k) are treated as d.

   We interpret a unary predicate P as follows:

   $$P^{\mathcal{A}'} = \begin{cases} P^{\mathcal{A}} & \text{if } d \notin P^{\mathcal{A}} \\ P^{\mathcal{A}} \cup D & \text{otherwise} \end{cases}$$

   The construction can be extended for n-ary predicates, by looking at each position separately.

   Similary, we can give the modified interpretation for a unary function symbol f:

   $$f^{\mathcal{A}'}(x) = \begin{cases} f^{\mathcal{A}}(x) & \text{if } x \notin D \\ f^{\mathcal{A}}(d) & \text{if } x = (d, k) \in D \end{cases}$$

   Extending to an infinite model works in exactly the same way, except for adding infinitely many copies of d by dropping the $k < m - n$ condition.

2. This does not work, because the $=$ predicate allows one to distinguish between different elements.

   Counterexample: The formula $F = \forall x \forall y (x = y)$ has a trivial model $\mathcal{A}$ with cardinality 1. Obviously, there cannot be any larger model.

Exercise 7.3. [Natural Numbers and FOL]
We consider the following axioms in an attempt to model the natural numbers in predicate logic:

1. $F_1 = \forall x \forall y (f(x) = f(y) \to x = y)$
2. $F_2 = \forall x (f(x) \neq 0)$
3. $F_3 = \forall x (x = 0 \vee \exists y (x = f(y)))$

Give a model with an *uncountable* universe for:

1. $\{F_1, F_2\}$
2. $\{F_1, F_2, F_3\}$

Hint: A set S is uncountable if there is no bijection between S and $\mathbb{N}$.

Solution:

1. $U_{\mathcal{A}} = \mathbb{R}_0^+$, $0^{\mathcal{A}} = 0$, and $f^{\mathcal{A}}(x) = x + 1$
   $f^{\mathcal{A}}$ is clearly injective and there is no x such that $f^{\mathcal{A}}(x) = 0$, because $-1 \notin U_{\mathcal{A}}$.
2. We take $U_{\mathcal{A}}$ to be the union of the positive real numbers and the non-positive whole numbers, i.e., $U_{\mathcal{A}} = \mathbb{R}_{>0} \cup \mathbb{Z}_{\leq 0}$.
   Let the symbols be interpreted as follows:
   $$0^{\mathcal{A}} = 0$$
   $$f^{\mathcal{A}}(x) = \begin{cases} 2x & \text{if } x > 0 \\ x - 1 & \text{if } x \leq 0 \end{cases}$$
   (a) $f^{\mathcal{A}}$ is defined as two disjoint domains that have disjoint ranges. Both domains are injective, hence the entire function is injective.
   (b) 0 is not in the range of $f^{\mathcal{A}}$: For $x > 0$, $f^{\mathcal{A}}(x) > 0$ and for $x \leq 0$, $f^{\mathcal{A}}(x) \leq -1$.
   (c) To show: $x \neq 0 \to \exists y (x = f(y))$.
   If $x < 0$, then $x \leq -1$, hence $x = f^{\mathcal{A}}(x + 1)$.
   Otherwise, $x = f^{\mathcal{A}}\left(\frac{x}{2}\right)$.

Homework 7.1. **[Invalid Herbrand Models]** (8 points)
Recall the fundamental theorem from the lecture: "Let F be a closed formula in Skolem form. Then F is satisfiable iff it has a Herbrand model".

Explain "what goes wrong" if the precondition is violated: when F is not closed or not in Skolem form. Describe both cases.

Homework 7.2. **[Proof of the Fundamental Theorem]** (6 points)
Recall the fundamental theorem: Let F be a closed formula in Skolem form. Then F is satisfiable iff it has a Herbrand model. Give the omitted proof for the base case (slide 6, $\mathcal{A}(G) = \mathcal{T}(G)$).

Homework 7.3. **[Herbrand Models]** (6 points)
Given the formula

$$F = \forall x(P(f(x)) \leftrightarrow \neg P(x))$$

1. Specify a Herbrand model for F.
2. Specify a Herbrand structure suitable for F, which is not a model of F.