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SS 2018 EXERCISE SHEET 9 05.06.2018

Submission of homework: Before tutorial on 12.06.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 9.1. [Decidability]

1. Resolution for first-order logic is sound and complete.
2. Satisfiability and validity for first-order logic are undecidable.

How do you reconcile these two facts?

Solution:
Resolution gives us a semi-decision procedure for unsatisfiability. That is, if a given formula is not unsatisfiable, it might not terminate. For it to be a decision procedure, it would need to always terminate.

Exercise 9.2. [Ground Resolution]
Use ground resolution to prove that the following formula is valid:

$$(\forall x P(x, f(x))) \longrightarrow \exists y P(c, y)$$

Solution:

$$\begin{aligned}
&\neg((\forall x P(x, f(x))) \longrightarrow \exists y P(c, y)) \\
&(\forall x P(x, f(x))) \wedge \neg \exists y P(c, y)) \\
&(\forall x P(x, f(x))) \wedge \forall y \neg P(c, y)) \\
&\forall x \forall y (P(x, f(x)) \wedge \neg P(c, y)) && \text{(Skolem-Form)}
\end{aligned}$$

Now enumerate the Herbrand expansion:

$$CE(F) = \{P(c, f(c)), \neg P(c, f(c)), \ldots\}$$

With resolution, we immediately get $\square$ from the first item in the enumeration.

Exercise 9.3. [Equality]
We consider how to model equality in predicate logic. In the lecture slides, the following axiom schema for congruence is used:

$$\frac{Eq(x_i, y)}{Eq(f(x_1, \ldots, x_i, \ldots, x_n), f(x_1, \ldots, y, \ldots, x_n))}$$

Assume that this schema is replaced by:

$$\frac{Eq(x_1, y_1) \qquad \cdots \qquad Eq(x_n, y_n)}{Eq(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n))}$$

Reflexivity, symmetry and transitivity stay unchanged. Show that the above modified schemas is equivalent to the schemas from the lecture.

Hint: Simulate the modified schema with the original one and vice versa.

Solution:
We first simulate the modified schema with the original one. Because the original schema only allows us to replace one term at a time, an induction is necessary. We want to prove $Eq(f(x_1, \ldots, x_n), f(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n))$ for $1 \leq m \leq n$. With $m = n$ we obtain the desired schema, hence the induction must proceed on m.

- Base case: $m = 1$

$$\frac{Eq(x_1, y_1)}{Eq(f(x_1, \ldots, x_n), f(y_1, x_2, \ldots, x_n))}$$

- Induction step: $m \rightsquigarrow m + 1$

$$\textsc{Trans}\ \frac{\textsc{IH}\ \dfrac{}{Eq(f(x_1, \ldots, x_n), f(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n))} \qquad \dfrac{Eq(x_{m+1}, y_{m+1})}{Eq(f(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n), f(y_1, \ldots, y_{m+1}, x_{m+2}, \ldots, x_n))}}{Eq(f(x_1, \ldots, x_n), f(y_1, \ldots, y_{m+1}, x_{m+2}, \ldots, x_n))}$$

Now, the opposite direction.

$$\frac{\dfrac{}{Eq(x_1, x_1)} \qquad \cdots \qquad Eq(x_i, y) \qquad \cdots \qquad \dfrac{}{Eq(x_n, x_n)}}{Eq(f(x_1, \ldots, x_i, \ldots, x_n), f(x_1, \ldots, y, \ldots, x_n))}$$

Homework 9.1. **[Restricted Resolution]** (8 points)
In the resolution procedure as defined in the lecture slides, we can unify arbitrarily many literals from two clauses. Consider a modified resolution procedure, where exactly one literal is picked. We add another rule ("collapsing rule"): For a clause $C = \{L_1, \ldots, L_n\}$, where $\{L_i, L_j\}$ can be unified using a mgu δ, add another clause $C' = (C - L_i)\delta$.

For example, given the clause

$$C = \{\neg W(x), \neg W(f(y)), T(x, y), \neg W(f(c))\}$$

we can apply the collapsing rule as follows:

$$L_1 = \neg W(x), L_2 = \neg W(f(y)), \delta = \{x \mapsto f(y)\}, C' = \{\neg W(f(y)), T(f(y), y), \neg W(f(c))\}$$

(Note that there are multiple possible ways to apply the collapsing rule to C.)

Prove that our modified resolution calculus, including collapsing rule, can be simulated by the original resolution calculus, and vice versa.

Homework 9.2. **[Resolution]** (6 points)
Show with resolution that:

$$f(f(f(a))) = a \longrightarrow f(f(a)) = a \longrightarrow f(a) = a$$

is valid. First, remove equality based on the procedure from the lecture. Then perform resolution.