Submission of homework: Before tutorial on 19.06.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 10.1. [$\exists^*\forall^*$ **with Equality**]
Show that unsatisfiability of formulas from the $\exists^*\forall^*$ fragment with equality is decidable.

Hint: Reduce it to the $\exists^*\forall^*$-fragment without equality.

Solution:
Applying the reduction of equality to non-equality from the lecture only inserts some (isolated) $\forall$-quantifiers, thus preserving the $\exists^*\forall^*$-fragment.

Exercise 10.2. [$\exists^*\forall^2\exists^*$]
Show how to reduce deciding unsatisfiability of formulas from the $\exists^*\forall^2\exists^*$-fragment to deciding unsatisfiability of formulas from the $\forall^2\exists^*$-fragment.

Solution:
Using skolemization for the outer existential quantifiers preserves satisfiability, and replaces variables by skolem constants, i.e., introduces no function symbols of arity > 0. The resulting formula is obviously in the $\forall^2\exists^*$-fragment.

Exercise 10.3. [**Finite Model Property**]
A set of formulas $\mathcal{F}$ is said to have the *finite model property* if for all $F \in \mathcal{F}$, the following two statements are equivalent:

1. F is satisfiable.
2. F has a finite model.

Give a decision procedure for satisfiability of any such set of formulas.

Solution:
Run the following routines in parallel:

1. Resolution (to check unsatisfiability).
2. Enumerate all finite models with universes $\{1, \dots, n\}$ (to check satisfiability).

If the formula F is unsatisfiable, resolution will terminate. The result of the decision procedure is "unsatisfiable". If it is satisfiable, resolution might not terminate, but because of the finite model property, F will have a finite model (that is isomorphic to a model with universe $\{1, \dots, n\}$) that will be enumerated eventually. The result is "satisfiable".

Exercise 10.4. [Sequent Calculus]
Prove the following formulas in sequent calculus:

1. $\neg\exists x P(x) \to \forall x \neg P(x)$
2. $(\forall x(P \lor Q(x))) \to (P \lor \forall x Q(x))$

Solution:

1.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(y) \Rightarrow \exists x P(x), P(y)}}{\Rightarrow P(y), \exists x P(x), \neg P(y)}\,\neg R}{\Rightarrow \exists x P(x), \neg P(y)}\,\exists R}{\Rightarrow \exists x P(x), \forall x \neg P(x)}\,\forall R}{\neg\exists x P(x) \Rightarrow \forall x \neg P(x)}\,\neg L}{\neg\exists x P(x) \to \forall x \neg P(x)}\,\to R
$$

2.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\forall x(P \lor Q(x))), P \Rightarrow P, Q(x)}\,Ax \qquad \dfrac{}{(\forall x(P \lor Q(x))), Q(x) \Rightarrow P, Q(x)}\,Ax}{(\forall x(P \lor Q(x))), (P \lor Q(x)) \Rightarrow P, Q(x)}\,\lor L}{\forall x(P \lor Q(x)) \Rightarrow P, Q(x)}\,\forall L}{\forall x(P \lor Q(x)) \Rightarrow P, \forall x Q(x)}\,\forall R}{\forall x(P \lor Q(x)) \Rightarrow P \lor \forall x Q(x)}\,\lor R}{\Rightarrow (\forall x(P \lor Q(x))) \to (P \lor \forall x Q(x))}\,\Longrightarrow R
$$

Exercise 10.5. [Miniscoping]
In the lecture, we proved that deciding unsatisfiability of monadic FOL formulas can be reduced to deciding unsatisfiability of formulas from the $\exists^*\forall^*$ fragment by using miniscoping.

Prove the lemma that after miniscoping, no nested quantifiers remain.

Solution:
We prove by induction on the structure of the formula that after miniscoping, for each subformula of the form $\forall x F$ resp. $\exists x F$, F is a disjunction resp. conjunction of literals, each literal containing x free.

The only interesting cases are the quantifier cases. Assume we have a formula of the form $\exists x F$, such that no miniscoping rules are applicable, and by induction hypothesis, below quantifiers in F there are only disjunctions/conjunctions of literals containing the bound variable.

As no miniscoping rules are applicable, F must be a conjunction of literals and quantified formulas, such that each conjunct contains x free. So assume F contains a quantified formula, i.e., $F = \ldots \land Qy.F' \land \ldots$. By induction hypothesis, F' is a disjunction/conjunction of literals, each literal containing y free. However, as we are in the monadic fragment, a literal can contain at most one free variable. Thus, F' cannot contain x free, which is a contradiction to F containing quantifiers. Thus, F only contains literals, and thus has the desired shape.

The case for $\forall x F$ is similar.

Homework 10.1. [FOL without Function Symbols] (8 points)
Describe an algorithm that transforms any formula (in FOL with equality) into an equisatisfiable formula (in FOL with equality) that does not use function symbols.

Hints: Functions can be modelled as relations satisfying some additional properties. Don't forget to deal with constants, i.e., functions with arity 0. A similar transformation as in the previous exercise might be helpful.

Homework 10.2. [Sequent Calculus] (6 points)
Prove the following statements using sequent calculus if they are valid, or give a countermodel otherwise.

1. $\neg\forall x \exists y \forall z(\neg P(x, z) \wedge P(z, y))$
2. $\forall x \forall y \forall z(P(x, x) \wedge (P(x, y) \wedge P(y, z) \rightarrow P(x, z)))$

Note: While you are free to carry out the sequent calculus proofs in Logitext, application of $\forall L$ and $\exists R$ delete the principal formula. You have to select "Contract" first before instantiating the principal formula.

Homework 10.3. [A Strange Island] (6 points)
You are visiting an island. It is inhabited by two kinds of people: *knights* and *knaves*. Knights always tell the truth and knaves always lie.

You interview all inhabitants. Every inhabitant tells you "we are all of one kind".

1. Model this situation as a formula in first-order logic.
2. Give a model that corroborates the story, or alternatively explain why it is contradictory. Use any calculus from the lecture for that (e.g. resolution).