Submission of homework: Before tutorial on 26.06.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 11.1. [Decidable Theories]

Let S be a set of sentences (i.e. closed formulas) such that S is closed under consequence: if $S \models F$ and F is closed, then $F \in S$. Additionally, assume that S is finitely axiomatizable and complete, i.e. $F \in S$ or $\neg F \in S$ for any sentence F.

1. Give a procedure for deciding, given only the axiomatization of S, whether $S \models F$ for a sentence F.
2. Can you obtain a similar result when the assumption is that the axiom system is only *recursively enumerable*?

Solution:

1. Let M be the set of axioms. Run resolution on $M \wedge F$ and $M \wedge \neg F$ in parallel. If $F \notin S$, then $M \wedge F \vdash \Box$ and the first resolution terminates. If $F \in S$, then $M \wedge \neg F \vdash \Box$ and the second resolution terminates.
2. Yes, by compactness. Enumerate all finite subsets of the axiom set and run resolutions in parallel.

Exercise 11.2. [Consequence]

Show that Cn is a closure operator, i.e. Cn fulfills the following properties:

- $S \subseteq Cn(S)$
- if $S \subseteq S'$ then $Cn(S) \subseteq Cn(S')$
- $Cn(Cn(S)) = Cn(S)$

Solution:

In the following, suppose S, S' are sets of Σ-sentences and F is Σ-sentence.

$$F \in S \Longrightarrow S \models F \Longrightarrow Cn(S) \models F$$

$$F \in Cn(S) \Longrightarrow S \models F \Longrightarrow S' \models F \Longrightarrow F \in Cn(S')$$

From the above two: $Cn(S) \subseteq Cn(Cn(S))$

$$F \in Cn(Cn(S)) \Longrightarrow Cn(S) \models F \Longrightarrow^{(*)} S \models F \Longrightarrow F \in Cn(S)$$

We have (*) because $\mathcal{A} \models Cn(S)$ iff $\mathcal{A} \models S$ by definition of Cn.

Exercise 11.3. [Axiomatizations and Compactness]
Using compactness, show that if a theory is finitely axiomatizable, any countable axiomatization of it has a finite subset that axiomatizes the same theory. In other words, if $Cn(\Gamma) = Cn(\Delta)$ with Γ countable and Δ finite, then there is a finite $\Gamma' \subseteq \Gamma$ with $Cn(\Gamma') = Cn(\Gamma)$.

Solution:
Claim: We can construct a finite subset $\Gamma' \subseteq \Gamma$ that axiomatizes $Cn(\Delta)$. In particular, $\Gamma' \vdash \Delta$ must hold. This is equivalent to $\Gamma', \neg\Delta \vdash \bot$.

We also know that $\Gamma, \neg\Delta \vdash \bot$, because Γ axiomatizes $Cn(\Delta)$. Hence, the infinite set of formulas $\Gamma \cup \{\neg\Delta\}$ is unsatisfiable. By compactness, there must be a finite subset that is unsatisfiable.

We can find this subset by enumerating all finite subsets $\Gamma' \subseteq \Gamma$ and running resolution on $\Gamma', \neg\Delta$.

Exercise 11.4. [Natural Deduction]
Prove the following formula using natural deduction.

$$\neg(\forall x(\exists y(\neg P(x) \land P(y))))$$

Solution:

$$\dfrac{\dfrac{\forall E\;\dfrac{[\forall x\exists y(\neg P(x)\land P(y))]}{\exists y(\neg P(x_1)\land P(y))} \qquad \dfrac{\dfrac{[\neg P(x_1)\land P(y_1)]}{P(y_1)}\land E_2 \qquad \dfrac{\forall E\;\dfrac{[\forall x\exists y(\neg P(x)\land P(y))]}{\exists y(\neg P(y_1)\land P(y))} \qquad \dfrac{[\neg P(y_1)\land P(y_2)]}{\neg P(y_1)}\land E_1}{\neg P(y_1)}\exists E}{\bot}\neg E}{\bot}\exists E}{\neg(\forall x(\exists y(\neg P(x)\land P(y))))}\neg I$$

Homework 11.1. **[Counterexamples from Sequent Calculus]** (4 points)
Consider the statement $\forall x P(x) \to \neg P(f(x))$.

1. What happens when trying to prove the validity of this formula in sequent calculus?
2. How can we derive a countermodel from the proof tree?
3. Is there a smaller countermodel?

Homework 11.2. **[Proofs]** (8 points)
Prove the following statements using natural deduction.

1. $\neg\forall x \exists y \forall z(\neg P(x, z) \wedge P(z, y))$
2. $\exists x(P(x) \to \forall x P(x))$

Homework 11.3. **[Elementary Classes]** (8 points)
In this exercise, we assume that all structures and formulas share the same signature Σ.

We define the operator $Mod(S)$ that returns the class of all structures that model a set of formulas S. In other words, $Mod(S)$ contains all $\mathcal{A}$ such that $\mathcal{A} \models S$.

A class of models M is said to be Δ-*elementary* if there is a set of formulas S such that $M = Mod(S)$. If S is just a singleton set, i.e. there is a formula F such that $S = \{F\}$, then M is *elementary.*

Prove:

1. A class of models M is elementary if and only if there is a *finite* set of formulas S such that $M = Mod(S)$.
2. If M is elementary and $M = Mod(S)$, there is a finite subset $S' \subseteq S$ such that $M = Mod(S')$.