Prof. Tobias Nipkow
Lars Hupel

Submission of homework: Before tutorial on 03.07.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 12.1. [Substitution in Sequent Calculus]
Prove that $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_G \Gamma[t/x] \Rightarrow \Delta[t/x]$, where, for a set of formulas Γ, we define $\Gamma[t/x]$ to be $\{F[t/x] \mid F \in \Gamma\}$, i.e. free occurrences of x are replaced by t. Give two different proofs:

1. A syntactic proof, transforming the proof tree of $\vdash_G \Gamma \Rightarrow \Delta$.
2. A semantic proof, using correctness and completeness of $\vdash_G$.

Solution:

1. Given a proof of $\Gamma \Rightarrow \Delta$, we can apply the substitution $x \mapsto p$ throughout the proof tree and obtain a valid proof of $\Gamma[x \mapsto p] \Rightarrow \Delta[x \mapsto p]$. Formally, this is an induction on the structure of the proof. Each case in the induction considers one proof rule or axiom, and we have to show that the new proof is still an instance of the axiom. This is trivial in all cases, since the rules are given as schemas that do only concern the structure of the formulas, which is not changed by the substitution.
2. We can also prove the claim by using soundness and completeness. Assume that $\Gamma = \{p_1, \ldots, p_n\}$, $\Delta = \{q_1, \ldots, q_m\}$ and $\Gamma \Rightarrow \Delta$ is derivable. By soundness we know that $p_1 \wedge \cdots \wedge p_n \implies q_1 \vee \cdots \vee q_m$ is a tautology. By the substitution lemma (Corollary 2.4) we know that substituting p for x yields a tautology again, and the completeness theorem asserts that the resulting sequent is provable.

Exercise 12.2. [QE for DLO]
Use the quantifier-elimination procedure for DLOs to check whether the following formula is a member of $Th(DLO)$:

$$\exists x \forall y \exists z ((x < y \lor z < x) \land y < z)$$

Solution:

$$\begin{aligned}
&\exists x \forall y \exists z ((x < y \lor z < x) \land y < z) \\
\longleftrightarrow\ &\exists x \forall y (\exists z (x < y \land y < z) \lor \exists z (z < x \land y < z)) \\
\longleftrightarrow\ &\exists x \forall y ((x < y \land \exists z (y < z)) \lor \exists z (z < x \land y < z)) \\
\longleftrightarrow\ &\exists x \forall y ((x < y \land \top) \lor \exists z (z < x \land y < z)) \\
\longleftrightarrow\ &\exists x \forall y (x < y \lor y < x) \\
\longleftrightarrow\ &\exists x \neg \exists y \neg (x < y \lor y < x) \\
\longleftrightarrow\ &\exists x \neg \exists y ((y < x \lor x = y) \land (x < y \lor x = y)) \\
\longleftrightarrow\ &\exists x \neg \exists y ((y < x \land x < y) \lor (y < x \land x = y) \lor (x = y \land x < y) \lor (x = y)) \\
\longleftrightarrow\ &\exists x \neg \exists y (\bot \lor \bot \lor \bot \lor (x = y)) \\
\longleftrightarrow\ &\exists x \neg \top \longleftrightarrow \bot
\end{aligned}$$

Exercise 12.3. **[Fourier–Motzkin Elimination]**

1. $\exists x \exists y(2 \cdot x + 3 \cdot y = 7 \wedge x < y \wedge 0 < x)$
2. $\exists x \exists y(3 \cdot x + 3 \cdot y < 8 \wedge 8 < 3 \cdot x + 2 \cdot y)$

Solution:

$$\begin{aligned}
&\exists x \exists y(2 \cdot x + 3 \cdot y = 7 \wedge x < y \wedge 0 < x)\\
\leftrightarrow\, &\exists x \left(\exists y \left(y = \frac{7}{3} - \frac{2}{3} \cdot x \wedge x < y\right) \wedge 0 < x\right)\\
\leftrightarrow\, &\exists x \left(x < \frac{7}{3} - \frac{2}{3} \cdot x \wedge 0 < x\right)\\
\leftrightarrow\, &\exists x \left(x < \frac{7}{5} \wedge 0 < x\right)\\
\leftrightarrow\, &0 < \frac{7}{5} \leftrightarrow \top
\end{aligned}$$

$$\begin{aligned}
&\exists x \exists y(3 \cdot x + 3 \cdot y < 8 \wedge 8 < 3 \cdot x + 2 \cdot y)\\
\leftrightarrow\, &\exists x \exists y \left(y < \frac{8}{3} - x \wedge 4 - \frac{3}{2} \cdot x < y\right)\\
\leftrightarrow\, &\exists x \left(4 - \frac{3}{2} \cdot x < \frac{8}{3} - x\right)\\
\leftrightarrow\, &\exists x \left(\frac{8}{3} < x\right) \leftrightarrow \top
\end{aligned}$$

Exercise 12.4. **[Ferrante–Rackoff Elimination]**

$$\exists x(\exists y(x = 2 \cdot y) \rightarrow (2 \cdot x \geq 0 \vee 3 \cdot x < 2))$$

Solution:

$$\begin{aligned}
&\exists x(\exists y(x = 2 \cdot y) \rightarrow (2 \cdot x \geq 0 \vee 3 \cdot x < 2))\\
\leftrightarrow\, &\exists x(\top \rightarrow (2 \cdot x \geq 0 \vee 3 \cdot x < 2))\\
\leftrightarrow\, &\exists x(2 \cdot x \geq 0 \vee 3 \cdot x < 2)\\
\leftrightarrow\, &\exists x \left(0 < x \vee x = 0 \vee x < \frac{2}{3}\right)\\
\leftrightarrow\, &\left(\top \vee \top \vee \left(0 < 0 \vee 0 = 0 \vee 0 < \frac{2}{3}\right) \vee \cdots\right)\\
\leftrightarrow\, &\top
\end{aligned}$$

Homework 12.1. **[Subtraction Logic]** (8 points)
We consider a fragment of linear arithmetic, in which atomic formulas only take the form $x - y \leq c$ for variables x and y, and $c \in \mathbb{Q}$.

For a finite set S of such difference constraints, we can define a corresponding inequality graph $G(V, E)$, where V is the set of variables of S, and E consists of all the edges (x, y) with weight c for all constraints $x - y \leq c$ of S. Show that the conjuction of all constraints from S is satisfiable iff G does not contain a negative cycle.

How can you use this theorem to obtain a procedure for deciding whether a formula is a member of this fragment?

Homework 12.2. **[Min, Max, Abs]** (6 points)

1. Show that $\mathrm{Th}(\mathbb{R}, 0, 1, <, =, +, \min, \max)$ is decidable, where min and max return the minimum and maximum of two values.
2. Show that $\mathrm{Th}(\mathbb{R}, 0, 1, <, =, +, \min, \max, |\cdot|)$ is decidable, where $|\cdot|$ is the absolute value.

Homework 12.3. **[Optimizing DLO]** (6 points)
DLO suffers from a heavy performance loss because after each step, a DNF needs to be reconstructed. We want to study an optimization that may avoid this under some circumstances.

Assume that we want to eliminate an $\exists x F$ where

- F is closed (except for x),
- F contains no negations and quantifiers, and
- there are only lower or only upper bounds for x in F.

Then, $\exists x F \equiv \top$. Prove correctness of this optimization.